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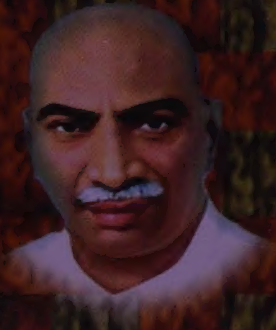
(University with Potential for Excellence)
DISTANCE EDUCATION



B.Sc.,

Mathematics
First Year
Paper - I

CALCULUS, THEORY OF EQUATION & TRIGONOMETRY

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MADURAI KAMARAJ UNIVERSITY

(The University with Potential for Excellence)

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B.Sc., Mathematics

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PAPER - 1

CALCULUS, THEORY OF EQUATION & TRIGONOMETRY

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Welcome

Dear students,

We welcome you as a student of the first year B.Sc., degree course.

This paper deals with the subject CALCULUS, THEORY OF EQUATIONS, and TRIGONOMETRY. The learning material for this paper will be supplemented by contact lectures.

In this book, first three units deals differential calculus and the later part of the fourth unit and fifth deals integral calculus and rest of the book deals theory of equations and trigonometry

Learning through the Distance Education mode, as you are all aware, involves self-learning and self-assessment and in this regard, you are expected to part in disciplined and dedicated effort.

As our part, we assure of our guidance and support.

Best Wishes

Department of Mathematics

SYLLABUS

B.Sc., First Year

Paper – I – Calculus, Theory of Equations and Trigonometry

CALCULUS

Unit I :

Successive differentiation – Expansion of functions – Leibinitz formula – Maxima and minima of functions of two variables.

Unit II :

Sub tangent and subnormal – polar coordinates – angle between the radius vector and the tangent – slope of the tangent – angle of intersection of two curves – polar sub tangent and polar subnormal – length of arc.

Unit III :

Envelopes – Curvatures – Circle, radius and centre of curvature – Evolutes.

Unit IV :

Polar coordinates – radius of curvature in polar coordinates – (p-r)equation – pedal equation of curves – Definite integrals and their properties.

Unit V :

Reduction formulae for $\sin^n x$, $\cos^n x$, $\tan^n x$, $\operatorname{cosec}^n x$, $\sec^n x$, $\cot^n x$, $\sin^n x \cos^m x$ – Bernoulli's formula – double and triple integral problems (Change of order of integration is excluded).

THEORY OF EQUATIONS & TRIGONOMETRY

Unit VI :

Theory of equations – Imaginary roots – Rational roots – Relation between the roots and coefficients – Symmetric functions of the roots.

Unit VII :

Sum of the powers of the roots of an equation – Newton's theorem – Transformation of equations – Roots multiplied by a given number – Reciprocal roots – reciprocal equations – Standard forms to increase and decrease the roots of a given equation by a given quantity.

Unit VIII :

Removal of terms – Descarte's rule of sign – Roll's theorem – Multiple roots – Strum's theorem – General solution of cubic equations – Cardon's method.

Unit IX :

Ferrari's method of solving biquadratic equation – Expansion of $\sin^n x$, $\cos^n x$, $\tan^x$, $\sin^n x \cos^m x$.

Unit X :

Hyperbolic function – Inverse Hyperbolic functions – Logarithm of complex numbers – Gregory series.

Text Books :

1. CALCULUS Vol I & II by T.K.Manikkavasagam Pillai and S. Narayanan
S.Viswanathan Publications
2. Algebra Vol I & II by T.K.Manikkavasagam Pillai and S.Narayanan
S.Viswanathan Publications
3. Trigonometry by T.K.Manikkavasagam Pillai and S.Narayanan
S.Viswanathan Publications, 11th edition, reprint 2006.

SCHEME OF LESSONS

PAPER – I

CALCULUS, THEORY OF EQUATIONS and TRIGONOMETRY

SCHEME OF LESSONS

UNIT I

SUCCESSIVE DIFFERENTIATION

Space for Hints

Space for Hints

Objectives :

In this unit, we are going to discuss successive differentiation, expansion of functions, Leibnitz formula, and Maxima and Minima of function of two variables.

- o After completing this unit, students may able to know
- o Higher derivatives
- o Expansion of functions
- o Applying Leibnitz formula for getting higher derivatives
- o Finding maxima or minima value of a function of two variables.

Introduction

The subject of Differential Calculus takes its stand upon the set of real numbers and concern itself with the various properties of the same. Moreover that subject of Differential Calculus had its origin mainly in the geometrical problem of the determination of the gradient of a curve as a point there of resulting in the determination of the tangent at the point. This subject has also deals the precise formulations of a large number of physical concepts such as velocity at an instant, acceleration at an instant, density at a point, specific heat at any temperature, etc., each of which appears as a Local or Instantaneous Rate of change as against the Average rate of change. Newton and Leibnitz first introduced the concept of Differential Calculus.

1.1. Successive Differentiation

In general if $y = f(x)$ then $\frac{dy}{dx}$ is also function of x.

Space for Hints

Thus, we may able to differentiate $\frac{dy}{dx}$ with respect to x and it is denoted by $\frac{d^2y}{dx^2}$ and it is called the *second derivative of y with respect to x.*

Hence $\boxed{\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)}$

Note : Second derivative of y with respect to x is denoted by $\frac{d^2y}{dx^2}$ or y'' or y_2 or $f''(x)$ or D^2y.

For certain functions, we may able to find the n^{th} derivative.

(i.e) if $y=f(x)$ then n^{th} derivative of y with respect to x is denoted by $\frac{d^ny}{dx^n}$ or $y^{(n)}$ or $f^{(n)}(x)$ or y_n of D^ny.

Example 1.1.1 :

Find y_n if $y=(a+bx)^m$.

Solution : Given that $y=(a+bx)^m$.

$$\therefore\ \frac{dy}{dx} = m(a+bx)^{m-1}.b$$

and $$\frac{d^2y}{dx^2} = m(m-1)(a+bx)^{m-2}.b^2$$

(i.e) $$\frac{d^2y}{dx^2} = b^2\cdot m(m-1)(a+bx)^{m-2}$$

and $$\frac{d^3y}{dx^3} = b^2\cdot m(m-1)(m-2)(a+bx)^{m-3}.b$$

(i.e) $$\frac{d^3y}{dx^3} = b^3\cdot m(m-1)(m-2)(a+bx)^{m-3}$$

Proceeding like above and after n differentiation, we have,

$$\frac{d^ny}{dx^n} = b^{n-1}\cdot m(m-1)(m-2)\mathrm{L}\ (m-(n-1))\cdot(a+bx)^{m-n}$$

(i.e) $$\frac{d^ny}{dx^n} = b^n\cdot m(m-1)(m-2)\mathrm{L}\ (m-(n-1))\cdot(a+bx)^{m-n}.$$

Space for Hints,

Note :

$$\frac{d^n}{dx^n}(a+bx)^{-1} = b^n(-1)(-2)(-3)\text{L} \;\; (-1n-+1)(a+bx)^{-1-n}$$

$$\text{(i.e)}\frac{d^n}{dx^n}(a+bx)^{-1} = (-1)^n b^n\, n!\cdot(a+bx)^{-1-n}$$

$$\boxed{(i.e) \quad \frac{d^n}{dx^n}(a+bx)^{-1} = \frac{(-1)^n b^n\, n!}{(a+bx)^{n+1}}.}$$

Example 1.1.2 :

Find y_n where $y=\log(a+bx)$.

Solution : Given that $y=\log(a+bx)$.

$$\therefore \;\; y_1 = \frac{1}{a+bx}\cdot b$$

$$\text{(i.e)}\; y_1 = b\cdot(a+bx)^{-1}$$

and $y_2 = b\cdot(-1)\cdot(a+bx)^{-2}\cdot b$

$$= b^2\cdot(-1)\cdot(a+bx)^{-2}$$

again $y_3 = b^2\cdot(-1)\cdot(-2)\cdot(a+bx)^{-3}\cdot b$

$$= b^3\cdot(-1)\cdot(-2)\cdot(a+bx)^{-3}$$

$$= b^3\cdot(-1)^2\cdot 2!\cdot(a+bx)^{-3}$$

and $y_4 = b^3\cdot(-1)\cdot(-2)\cdot(-3)\cdot(a+bx)^{-4}\cdot b$

$$= b^4\cdot(-1)^3\cdot 3!\cdot(a+bx)^{-4}$$

Hence proceeding like above, we get,

$$y_n = b^n\cdot(-1)^{n-1}\cdot(n-1)!\cdot(a+bx)^{-n}.$$

Note :

$$\boxed{\text{If } y=\log(a+bx) \text{ then } \frac{d^n y}{dx^n} = b^n\cdot(-1)^{n-1}\cdot(n-1)!\cdot(a+bx)^{-n}}$$

Space for Hints

Example 1.1.3 :

Find the n^{th} derivative of $\cos(a+bx)$.

Solution : Let $y = \cos(a+bx)$

$$\therefore \; y_1 = -\sin(a+bx)\cdot b$$

$$= -b\cdot\sin(a+bx)$$

$$= b\cdot\cos\left(\frac{\pi}{2}+a+bx\right) \quad \left\{\text{Q}\ \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta\right\}$$

$$\therefore \; y_2 = -b\cdot\sin\left(\frac{\pi}{2}+a+bx\right)\cdot b$$

$$= b^2\cdot\cos\left(\frac{\pi}{2}+\frac{\pi}{2}+a+bx\right)$$

$$= b^2\cdot\cos\left(2\cdot\frac{\pi}{2}+a+bx\right)$$

and $\therefore \; y_3 = -b^2\cdot\sin\left(\frac{\pi}{2}+a+bx\right)\cdot b$

$$= b^3\cdot\cos\left(\frac{\pi}{2}+2\cdot\frac{\pi}{2}+a+bx\right)$$

$$= b^3\cdot\cos\left(3\cdot\frac{\pi}{2}+a+bx\right)$$

Proceeding like above, we get,

$$y_n = b^n\cdot\cos\left(n\cdot\frac{\pi}{2}+a+bx\right)$$

(i.e) $y_n = b^n\cdot\cos\left(\frac{n\pi}{2}+a+bx\right)$

Note :

(1) If $y = \cos ax$ then $y_n = \cos\left(\frac{n\pi}{2}+ax\right)$

(2) If $y = \cos x$ then $y_n = \cos\left(\frac{n\pi}{2}+x\right)$

Space for Hints

Example 1.1.4. :

Find $\frac{d^n y}{dx^n}$ if $y = e^{ax}\sin(bx+c)$

Solution : Given that $y = e^{ax}\sin(bx+c)$.

$\therefore\ y_1 = e^{ax}(a)\sin(bx+c) + e^{ax}\cos(bx+c)(b)$

(i.e) $y_1 = e^{ax}(a\cdot\sin(bx+c) + b\cdot\cos(bx+c))$ ---------------------------- (1.1)

Put $a = r\cos\theta$, $b = r\sin\theta$

$\therefore\ a^2+b^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$\Rightarrow\ a^2+b^2 = r^2$

$\Rightarrow\ r = \sqrt{a^2+b^2}$

and $\frac{r\sin\theta}{r\cos\theta} = \frac{b}{a}$

$\Rightarrow\ \tan\theta = \frac{b}{a}$

$\Rightarrow\ \theta = \tan^{-1}\left(\frac{b}{a}\right)$

$\therefore$ (1.1) $\Rightarrow\ y_1 = e^{ax}(r\cos\theta\cdot\sin(bx+c) + r\sin\theta\cdot\cos(bx+c))$

(i.e) $y_1 = re^{ax}(\sin(\theta+bx+c))$

(i.e) $y_1 = re^{ax}(\sin(bx+c+\theta))$

(i.e) y_1 can obtained from y by multiplying r with y and angle increasing the angle by θ.

Similarly, $y_2 = r\cdot re^{ax}(\sin(bx+c+\theta+\theta))$

(i.e) $y_2 = r^2\cdot e^{ax}(\sin(bx+c+2\theta))$

Similarly, $y_3 = r\cdot r^2\cdot e^{ax}(\sin(bx+c+2\theta+\theta))$

(i.e) $y_3 = r^3\cdot e^{ax}(\sin(bx+c+3\theta))$

Proceeding like above we get,

$y_n = r^n\cdot e^{ax}(\sin(bx+c+n\theta))$.

Thus if $y = e^{ax}\sin(bx+c)$ then $y_n = r^n\cdot e^{ax}(\sin(bx+c+n\theta))$

where $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

Space for Hints

Check your progress

Question :

(1) If $y=\sin(ax+b)$ then find $D^n y$.

(2) If $y=\sin x$ then prove that $y_n = \sin\left(\frac{n\pi}{2}+x\right)$.

(3) Prove that $D^n\left\{e^{ax}\cos(bx+c)\right\} = r^n \cdot e^{ax}(\cos(bx+c+n\theta))$

where $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$.

Example 1.1.5 :

Find the n^{th} derivative of $\sin x \cdot \sin 3x$

Solution : Let $y=\sin x \cdot \sin 3x$

We know that $2\sin A \sin B = \cos(A-B)-\cos(A+B)$

$$\therefore \sin x \cdot \sin 3x = \frac{1}{2}[\cos(3x-x)-\cos(3x+x)]$$

$$= \frac{1}{2}[\cos(2x)-\cos(4x)]$$

Thus $y_n = \frac{1}{2}\left[2^n \cdot \cos(2x+n\frac{\pi}{2})-4^n \cdot \cos(4x+n\frac{\pi}{2})\right]$.

Example 1.1.6 :

Find y_n if $\sin^3 x \cdot \cos^2 x$

Solution : Let $y = \sin^3 x \cdot \cos^2 x$

(i.e) $y = \sin^2 x \cdot \cos^2 x \cdot \sin x$

(i.e) $y = (\sin x \cdot \cos x)^2 \cdot \sin x$

(i.e) $y = \left(\frac{2}{2}\sin x \cdot \cos x\right)^2 \cdot \sin x$

(i.e) $y = \frac{1}{4}\cdot(2\sin x \cdot \cos x)^2 \cdot \sin x$

(i.e) $y = \frac{1}{4}\cdot(\sin 2x)^2 \cdot \sin x$

Space for Hints

(i.e) $y = \frac{1}{4}\cdot\left(\frac{1}{2}\cdot(1-\cos 4x)\right)\cdot\sin x$ ($\because$ $\cos 2A = 1-2\sin^2 A$

$\Rightarrow 2\sin^2 A = 1-\cos 2A$

$\Rightarrow 2\sin^2 2x = 1-\cos 4x$)

(i.e) $y = \frac{1}{8}\cdot(1-\cos 4x)\cdot\sin x$

(i.e) $y = \frac{1}{8}\cdot(\sin x - \cos 4x\cdot\sin x)$

(i.e) $y = \frac{1}{16}\cdot(2\sin x - 2\cos 4x\cdot\sin x)$

(i.e) $y = \frac{1}{16}\cdot(2\sin x - (\sin(4x+x) - \sin(4x-x))$

($\because$ $2\cos A\cdot\sin B = \sin(A+B) - \sin(A-B)$)

(i.e) $y = \frac{1}{16}\cdot(2\sin x - \sin 5x + \sin 3x)$

(i.e) $y = \frac{1}{16}\cdot(2\sin x + \sin 3x - \sin 5x)$

Hence $y_n = \frac{1}{16}\left[2\cdot\sin\left(x+\frac{n\pi}{2}\right) + 3^n\cdot\sin\left(3x+\frac{n\pi}{2}\right) - 5^n\cdot\sin\left(5x+\frac{n\pi}{2}\right)\right]$.

Example 1.1.7 :

Find $D^n(\cos x\cos 2x\cos 3x)$

Solution : Let $y = \cos x\cos 2x\cos 3x$

(i.e) $y = \cos 3x\cos x\cos 2x$

$= \frac{1}{2}\left[2\cdot\cos 3x\cdot\cos x\right]\cdot\cos 2x$

$= \frac{1}{2}\left[\cos(3x+x) + \cos(3x-x)\right]\cdot\cos 2x$

($\because$ $2\cos A\,\cos B = \cos(A+B) + \cos(A-B)$)

$= \frac{1}{2}\left[\cos 4x + \cos 2x\right]\cdot\cos 2x$

$= \frac{1}{2}\left[\cos 4x\cdot\cos 2x + \cos^2 2x\right]$

$$= \frac{1}{4} \cdot 2 \cdot \left[\cos 4x \cdot \cos 2x + \cos^2 2x \right]$$

$$= \frac{1}{4} \cdot \left[2 \cdot \cos 4x \cdot \cos 2x + 2 \cdot \cos^2 2x \right]$$

$$= \frac{1}{4} \cdot \left[\cos 6x + \cos 2x + \cos 4x + 1 \right]$$

$$(\because \cos 2A = 2\cos^2 A - 1 \Rightarrow 2\cos^2 A = \cos 2A + 1)$$

Hence $y_n = 6^n \cos\left(6x + \frac{n\pi}{2}\right) + 4^n \cos\left(4x + \frac{n\pi}{2}\right) + 2^n \cos\left(2x + \frac{n\pi}{2}\right)$

Example 1.1.8 :

Find y_n if $\frac{1}{4x^2 + 8x + 3}$.

Solution : Let $y = \frac{1}{4x^2 + 8x + 3}$.

Now $y = \frac{1}{4x^2 + 8x + 3}$

$$= \frac{1}{(2x+1)(2x+3)}$$

Let $\frac{1}{(2x+1)(2x+3)} = \frac{A}{2x+1} + \frac{B}{2x+3}$.

(i.e) $1 = A(2x+3) + B(2x+1)$ -- (1.2)

put $x = -\frac{1}{2}$ in (1.2), we get,

$A\left(2\left(-\frac{1}{2}\right)+3\right) = 1$

(i.e) $A(2) = 1$

(i.e) $A = \frac{1}{2}$

Again put $x = -\frac{3}{2}$ in (1.2), we get,

$B(-3+1) = 1$

Space for Hints

Space for Hints

(i.e) $B = -\frac{1}{2}$

$$\therefore\ y = \frac{1/2}{2x+1} + \frac{-1/2}{2x+3}$$

(i.e) $y = \frac{1}{2}\cdot\frac{1}{2x+1} - \frac{1}{2}\cdot\frac{1}{2x+3}$.

Thus $y_n = \frac{1}{2}\left[\frac{(-1)^n \cdot n!\cdot 2^n}{(2x+1)^{n+1}}\right] - \frac{1}{2}\left[\frac{(-1)^n \cdot n!\cdot 2^n}{(2x+3)^{n+1}}\right]$

Example 1.1.9 :

If $y = \frac{x^2}{(2x+1)^2(x+2)}$.

Solution : Given that $y = \frac{x^2}{(2x+1)^2(x+2)}$.

Let $\frac{x^2}{(2x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$

(i.e) $x^2 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$ ---------------------- (1.3)

put $x = -1$ in (1.3), we get,

$B(-1+2) = (-1)^2$

$\Rightarrow\ B = 1$

Put $x = -2$ in (1.3), we get,

$C(-1)^2 = (-2)^2$

$\Rightarrow\ C = 4$

put $x = 0$ in (1.3), we get,

$A(1)(2) + B(2) + C(1) = 0$

$\Rightarrow\ 2A + 2B + C = 0$

$\Rightarrow\ 2A + 2(1) + 4 = 0$

$\Rightarrow\ 2A = -6$

$\Rightarrow\ A = -3$

$$y = \frac{-3}{x+1} + \frac{1}{(x+1)^2} + \frac{4}{x+2}$$

Space for Hints

Hence $y_n = 4\left[\frac{(-1)^n \cdot n!}{(x+2)^{n+1}}\right]+\left[\frac{(-1)^n \cdot (n+1)!}{(x+1)^{n+2}}\right]-3\left[\frac{(-1)^n \cdot n!}{(x+1)^{n+1}}\right]$

Note :

(1) $\frac{d^n}{dx^n}\left(\frac{1}{(a+bx)^2}\right)=\frac{(-1)^n \cdot (n+1)!}{(a+bx)^{n+2}}$

(2) $\frac{d^n}{dx^n}\left(\frac{1}{(a+bx)^m}\right)=\frac{(-1)^n \cdot (m+n+1)!}{(m-1)! \cdot (a+bx)^{m+n}}$

Example 1.1.10 :

If $y^3-3ax^2+x^3=0$, prove that $\frac{d^2y}{dx^2}+\frac{2a^2x^2}{y^5}=0$.

Proof : Given that $y^3-3ax^2+x^3=0$.

Differentiate with respect to x, we get,

$$3y^2\frac{dy}{dx}-6ax+3x^2=0$$

(i.e) $y^2\frac{dy}{dx}=2ax-x^2$

(i.e) $\frac{dy}{dx}=\frac{2ax-x^2}{y^2}$

Again differentiate with respect to x, we get,

$$\frac{d^2y}{dx^2}=\frac{y^2(2a-2x)-(2ax-x^2)\cdot 2y\cdot\frac{dy}{dx}}{y^4}$$

$$=\frac{y^2(2a-2x)-(2ax-x^2)\cdot 2y\cdot\left(\frac{2ax-x^2}{y^2}\right)}{y^4}$$

$$=\frac{2y^3(a-x)-2(2ax-x^2)^2}{y^5}$$

$$=\frac{2\cdot(3ax^2-x^3)(a-x)-2(2ax-x^2)^2}{y^5}$$

$(\because\ y^3=3ax^2-x^3)$

Space for Hints

$$= \frac{2\cdot(3a^2x^2 - \cancel{3ax^3} - \cancel{ax^3} + \cancel{x^4} - 4a^2x^2 - \cancel{x^4} + \cancel{4ax^3})}{y^5}$$

$$= \frac{2(-a^2x^2)}{y^5}$$

(i.e) $\dfrac{d^2y}{dx^2} + \dfrac{2a^2x^2}{y^5} = 0$.

This proves the problem.

Example 1.1.11 :

If $x^3 + y^3 - 3axy = 0$, prove that $\dfrac{d^2y}{dx^2} = \dfrac{2a^2xy}{(ax-y^2)^3}$.

Proof : Given that $x^3 + y^3 - 3axy = 0$.

Differentiate with respect to x, we get,

$$3x^2 + 3y^2\frac{dy}{dx} - 3a\left(x\frac{dy}{dx} + y\cdot 1\right) = 0.$$

Dividing by 3 on both sides, we get,

$$x^2 + y^2\frac{dy}{dx} - ax\frac{dy}{dx} - ay = 0.$$

(i.e) $\dfrac{dy}{dx}(y^2 - ax) = ay - x^2$

(i.e) $\dfrac{dy}{dx} = \dfrac{ay - x^2}{y^2 - ax}$

Again differentiate with respect to x, we get,

(i.e) $\dfrac{d^2y}{dx^2} = \dfrac{\left(y^2 - ax\right)\cdot\left(a\dfrac{dy}{dx} - 2x\right) - \left(ay - x^2\right)\cdot\left(2y\dfrac{dy}{dx} - a\right)}{\left(y^2 - ax\right)^2}$

(i.e) $\dfrac{d^2y}{dx^2} = \dfrac{\left(y^2 - ax\right)\cdot\left(a\dfrac{ay - x^2}{y^2 - ax} - 2x\right) - \left(ay - x^2\right)\cdot\left(2y\dfrac{ay - x^2}{y^2 - ax} - a\right)}{\left(y^2 - ax\right)^2}$

(i.e) $\dfrac{d^2y}{dx^2} = \dfrac{\left(y^2 - ax\right)\cdot\left(a\dfrac{ay - x^2}{y^2 - ax} - 2x\right) - \left(ay - x^2\right)\cdot\left(2y\dfrac{ay - x^2}{y^2 - ax} - a\right)}{\left(y^2 - ax\right)^2}$

Space for Hints

$$= \frac{-2xy^4 + 6ax^2y^2 - 2a^3xy - 2x^4y}{\left(y^2 - ax\right)^3}.$$

$$= \frac{-2xy(y^4 - 3axy + x^3) - 2a^3xy}{\left(y^2 - ax\right)^3}.$$

$$= \frac{-2xy(0) - 2a^3xy}{\left(y^2 - ax\right)^3}.$$

$$= \frac{-2a^3xy}{\left(y^2 - ax\right)^3}.$$

$$= \frac{2a^2xy}{(ax - y^2)^3}.$$

Therefore $\dfrac{d^2y}{dx^2} = \dfrac{2a^2xy}{(ax - y^2)^3}$.

This proves the problem.

Example 1.1.12. :

If $y = e^{-x}\cos x$, prove that $\dfrac{d^4y}{dx^4} + 4y = 0$

Proof : Given that $y = e^{-x}\cos x$.

Differentiate with respect to x, we get,

$$\frac{dy}{dx} = e^{-x} \cdot (-\sin x) + (-e^{-x})\cos x.$$

Again differentiate with respect to x, we get,

$$\frac{d^2y}{dx^2} = e^{-x}(-\cos x) + (-\sin x)(-e^{-x}) + e^{-x}\cos x + (-e^{-x})(-\sin x)$$

$$= -\cancel{e^{-x}\cos x} + e^{-x} \cdot \sin x + \cancel{e^{-x}\cos x} + e^{-x}\sin x$$

$$= 2e^{-x} \cdot \sin x$$

(i.e) $\dfrac{d^2y}{dx^2} = 2e^{-x} \cdot \sin x$.

Again differentiate with respect to x, we get,

Space for Hints

$$\frac{d^3y}{dx^3} = 2\cdot e^{-x}(\cos x) + 2(e^{-x})(\sin x)$$

and $$\frac{d^4y}{dx^4} = 2\cdot e^{-x}(\cos x) + 2(e^{-x})(-\sin x)$$

$$+\ 2\cdot e^{-x}(\sin x) + 2(e^{-x})(\cos x).$$

$$= -2\cdot e^{-x}\cos x - 2\cdot e^{-x}\cos x$$

$$= -4\cdot e^{-x}\cos x$$

(i.e) $$\frac{d^4y}{dx^4} + 4y = 0.$$

This proves the problem.

Example 1.1.13 :

If $y = (A+Bx)e^{kx}$, show that $\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + k^2y = 0$.

Proof : Given that $y = (A+Bx)e^{kx}$.

Differentiate with respect to x, we get,

$$\frac{dy}{dx} = (A+Bx)e^{kx}(k) + B\cdot e^{kx}$$

(i.e) $$\frac{dy}{dx} = ky + B\cdot e^{kx} \qquad \text{------------------------------} \quad (1.4)$$

Differentiate with respect to x, we get,

$$\frac{d^2y}{dx^2} = k\frac{dy}{dx} + B\cdot k\cdot e^{kx}$$

(i.e) $$\frac{d^2y}{dx^2} = k\frac{dy}{dx} + k\cdot\left(\frac{dy}{dx} - ky\right) \text{ (from (1.4))}$$

(i.e) $$\frac{d^2y}{dx^2} = k\frac{dy}{dx} + k\cdot\frac{dy}{dx} - k^2y$$

(i.e) $$\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + k^2y = 0.$$

This proves the problem.

Space for Hints

Check your progress

Question :

If $y = a\cos(\log x) + b\sin(\log x)$, prove that $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$.

1.2. Expansions of functions

The series $a_1 + a_2 + a_3 + \text{L}$ is called an infinite series and is denoted by $\sum_{n=1}^{\infty} a_n$ or simply $\sum a_n$ and $a_1 + a_2 + a_3 + \text{L} + a_n$ is called sum of first n terms of the series and it is denoted by S_n.

(i.e) $S_n = a_1 + a_2 + a_3 + \text{L} + a_n$.

When n tends to infinity (i.e $n \to \infty$) then S_n may tend to

(i) a finite limit (or)

(ii) infinity (or)

(iii) more than one limit.

Now

(i) $\lim_{n\to\infty} S_n$ = finite then we say $\sum a_n$ is convergent

(ii) $\lim_{n\to\infty} S_n = \infty$ finite then we say $\sum a_n$ is divergent and

(iii) $\lim_{n\to\infty} S_n$ is not a unique finite limit,

(i.e) more than one limit, then $\sum a_n$ is said to be an oscillating series.

1.2.1 Cauchy form of Remainder

Cauchy form of remainder of a function $f(x)$ is given by

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \text{L} + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) R_n$$

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where $R_n = \frac{(x-a)^n (1-\theta)^{n-1}}{(n-1)!} f^{(n)}(a+(x-a)\theta)$; $0<\theta<1$.

(i.e) $f(x) = S_n + R_n$ where

$$S_n = f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\text{L} + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)$$

and $R_n = \frac{(x-a)^n (1-\theta)^{n-1}}{(n-1)!} f^{(n)}(a+(x-a)\theta)$; $0<\theta<1$.

1.2.2 Taylor's infinite series

If a function $f(x)$ possesses derivatives of all orders in the interval $(a, a+h)$, then for every integer n, however large, there corresponds a Taylor's development with Cauchy form of remainder and the Taylor's series is given by

$$f(x) = f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\text{L}$$

If we consider $a=0$ then the resultant series is called Maclaurin's series.

Thus the Maclaurin's series of $f(x)$ is given by

$$f(x) = f(0)+x f'(0)+\frac{x^2}{2!}f''(0)+\text{L}$$

Note :

Maclaurin's series is useful in finding the expansion of functions.

Example 1.2.1 :

Find the expansion of $\cos x$.

Solution : Let $f(x) = \cos x$.

Now $f(x) = \cos x$.

$\therefore \quad f'(x) = -\sin x$,

$f''(x) = -\cos x$,

$f'''(x) = \sin x$,

$f^{(iv)}(x) = \cos x$,

$f^{(v)}(x) = -\sin x$,

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$f^{(vi)}(x) = -\cos x$ and so on.

$\therefore\ f(0) = \cos 0 = 1,$

$f'(0) = 0,$

$f''(0) = -1,$

$f'''(0) = 0,$

$f^{(iv)}(0) = 1.$

$f^{(v)}(0) = 0,$

$f^{(vi)}(0) = -1$ and so on.

Now $R_n(x) = \dfrac{x^n}{n!} f^{(n)}(\theta x)$, $0 < \theta < 1$.

We know that if $f(x) = \cos x$ then

$$f^{(n)}(x) = \cos\left(\frac{n\pi}{2} + x\right).$$

Now $\left|\cos\left(\dfrac{n\pi}{2} + x\right)\right| \le 1$ and $\lim\limits_{n\to\infty} \dfrac{x^n}{n!} = 0$, then we have

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty}\left(\frac{x^n}{n!}\cos\left(\frac{n\pi}{2} + x\right)\right) = 0$$

Hence by Maclaurin's series, we have

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \mathrm{L}$$

Example 1.2.2 :

Find the expansion of $\tan x$.

Proof : Let $f(x) = \tan x$.

Now $f(x) = \tan x$.

$\therefore\ y_1 = f'(x) = \sec^2 x$

$= 1 + \tan^2 x$

$= 1 + y^2$

(i.e) $y_1 = 1 + y^2$.

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$\therefore\ y_2 = 2yy_1$

and $y_3 = 2yy_2 + 2(y_1)^2$

$= 2yy_2 + 2y_1^2$

and $y_4 = 2yy_3 + 2y_1y_2 + 4y_1y_2$

$= 2yy_3 + 6y_1y_2$

and $y_5 = 2yy_4 + 2y_1y_3 + 6y_1y_3 + 6y_2y_2$

$= 2yy_4 + 8y_1y_3 + 6y_2^2$

and so on.

Now $y(0) = f(0) = \tan 0 = 0$

and $y_1(0) = 1 + y^2(0) = 1 + 0 = 1,$

$y_2(0) = 2\,(0)\,(1) = 0,$

$y_3(0) = 0 + 2(1)^2 = 2,$

$y_4(0) = 2\,(0)\,(2) + 6\,(1)\,(0) = 0,$

$y_5(0) = 2\,(0)\,(6) + 8\,(1)\,(2) = 16,$

and so on.

$\therefore$ By Maclaurin's series,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \text{L}$$

$$\text{(i.e)}\ \tan x = 0 + x(1) + 0 + \frac{x^3}{3!}(2) + 0 + \frac{x^5}{5!}(16) + \text{L}$$

$$\text{(i.e)}\ \tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \text{L}$$

Example 1.2.3 :

Expand $\log(1+x)$ [Here log for natural logarithm]

Solution : Let $f(x) = \log(1+x)$

Now $f(x) = \log(1+x)$

$$\therefore\ f'(x) = \frac{1}{1+x},$$

$$\text{and}\ f''(x) = \frac{-1}{(1+x)^2},$$

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$$f'''(x) = \frac{(-1)(-2)}{(1+x)^3} = \frac{(-1)^2\, 2!}{(1+x)^3}$$

$$f^{(\text{iv})}(x) = \frac{(-1)^2\, 2!(-3)}{(1+x)^4} = \frac{(-1)^3\, 3!}{(1+x)^4}$$

$$f^{(\text{v})}(x) = \frac{(-1)^2\, 3!(-4)}{(1+x)^5} = \frac{(-1)^4\, 4!}{(1+x)^5}$$

$$\vdots \qquad \vdots \qquad \vdots$$

In general $f^{(n)}(x) = \dfrac{(-1)^{n-1}\,(n-1)!}{(1+x)^n}$

Now $f(0) = 0$,

$f'(0) = 1$,

$f''(0) = -1$,

$f'''(0) = 2!$

$f^{(\text{iv})}(0) = -3!$

$f^{(\text{v})}(0) = 4!$

$\vdots \quad \vdots \quad \vdots$

and $f^{(n)}(\theta x) = \dfrac{(-1)^{n-1}\,(n-1)!}{(1+\theta x)^n}$

Now
$$R_n(x) = \frac{x^n}{n!} \cdot f^{(n)}(\theta x)$$
$$= \frac{x^n}{n!} \cdot \frac{(-1)^{n-1}\,(n-1)!}{(1+\theta x)^n}$$
$$= \frac{(-1)^{n-1}\, x^n}{n \cdot (1+\theta x)^n}$$
$$= (-1)^{n-1} \cdot \frac{1}{n \cdot}\left(\frac{x}{1+\theta x}\right)^n$$

Now $R_n(x) = (-1)^{n-1} \cdot \dfrac{x^n}{1+\theta x \cdot}\left(\dfrac{1}{1+\theta x}\right)^{n-1}$.

If $|x| < 1$ then $\dfrac{1}{1+\theta x} < 1$.

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$$\therefore \lim_{n\to\infty}\left(\frac{1}{1+\theta x}\right)^{n-1} = 0.$$

Again if $|x|<1$ then $\lim_{n\to\infty} x^n = 0$.

Hence $\lim_{n\to\infty} R_n(x) = 0$.

Thus if $|x|<1$, $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$

Example 1.2.4 :

Expand $(1+x)^m$.

Solution : Let $f(x) = (1+x)^m$

$\therefore\ f'(x) = m(1+x)^{m-1}$,

and $f''(x) = m(m-1)(1+x)^{m-2}$,

$$f'''(x) = m(m-1)(m-2)(1+x)^{m-3},$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$f^{(n)}(x) = m(m-1)\cdots(m-n+1)(1+x)^{m-n}$$

Now $f^{(n)}(\theta x) = m(m-1)\cdots(m-n+1)(1+\theta x)^{m-n}$

Clearly the series $1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots$ is a divergent series for $|x| \le 1$

Now $R_n(x) = mx(1+\theta x)^{m-1}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \frac{(m-1)(m-2)\cdots(m-n+1)}{(n-1)!}x^{n-1}$

If $|x|<1$ then $\frac{1-\theta}{1+\theta x} < 1$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{1-\theta}{1+\theta x}\right)^n = 0.$$

Further we know that $\sum a_n$ is a convergent series then $(a_n) \to 0$ as $n \to \infty$

Now $\frac{(m-1)(m-2)\cdots(m-n+1)}{(n-1)!}$ is the n^{th} term of the convergent series

$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$ and therefore it is zero.

Thus $\lim_{n\to\infty} R_n(x) = 0$.

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Hence $(1+x)^m = 1 + mx + \dfrac{m(m-1)}{2!}x^2 + \dfrac{m(m-1)(m-2)}{3!}x^3 + \text{L}$ for $|x|<1$.

Check your progress

Questions :

1) Prove that $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \text{L}$

2) Prove that $e^{-x} = 1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \text{L}$

Example 1.2.5 :

Expand $\cosh x$.

Solution :

Step 1 : First we find the expansion of e^x.

Let $f(x) = e^x$

Now $f(x) = e^x$

$\therefore\ f'(x) = e^x$,

and $f''(x) = e^x$,

$f'''(x) = e^x$,

$f^{(\text{iv})}(x) = e^x$ and so on.

Now $f(0) = e^0 = 1$

and $f'(0) = e^0 = 1$,

$f''(0) = e^0 = 1$,

$f'''(0) = e^0 = 1$,

$f^{(\text{iv})}(0) = e^0 = 1$ and so on.

By Maclaurin's series, we get,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \text{L}$$

(i.e) $e^x = 1 + \dfrac{x}{1!}(1) + \dfrac{x^2}{2!}(1) + \text{L}$

(i.e) $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \text{L}$

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Step 2 : Using step 1, we have,

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + L$$

Step 3 : Now $\cosh x = \dfrac{e^x = e^{-x}}{2}$

$$= \frac{1}{2}\left(2 \div \frac{2x^2}{2!} + \frac{2x^4}{4!} + L \right)$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + L$$

Example 1.2.6 :

Prove that $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + L$ and deduce the value of π.

Solution : Let $y = \tan^{-1} x$

$$\therefore \ y_1 = \frac{1}{1+x^2} = (1+x^2)^{-1}.$$

$= 1 - x^2 + x^4 - x^6 + L$ using binomial theorem for $|x| < 1$.

$$\therefore \ y_2 = -2x + 4x^3 - 6x^5 + L$$

and $y_3 = -2 + 12x^2 - 30x^4 + L$,

$y_4 = 24x - 120x^3 + L$,

$y_5 = 24 - 360x^2 + L$ and so on.

Now $y(0) = \tan^{-1}(0) = 0$

$y_1(0) = 1$

$y_2(0) = 0$

$y_3(0) = -2$

$y_4(0) = 0$

$y_5(0) = 24$ and so on.

Using Maclaurin's series, we get,

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + L$$

(i.e) $y = y(0) + \dfrac{x}{1!} y_1(0) + \dfrac{x^2}{2!} y_2(0) + L$

Space for Hints

(i.e) $\tan^{-1} x = x - \frac{2x^3}{3!} + \frac{24x^5}{5!} - \text{L}$

(i.e) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \text{L}$

Deduction : put $x = 1$ in the expansion of $\tan^{-1} x$, we get,

$\tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \text{L}$

(i.e) $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \text{L}$

(i.e) $\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \text{L}\right)$.

Example 1.2.7 :

Expand $e^{\sin x}$.

Solution : Let $y = f(x) = e^{\sin x}$

Now $y = f(x) = e^{\sin x}$

$\therefore\ y_1 = \cos x\, e^{\sin x} = y\cos x$,

and $y_2 = y_1 \cos x - y \sin x$,

$y_3 = (y_2 \cos x - y_1 \sin x) - (y_1 \sin x + y\cos x)$

$= y_2 \cos x - y_1 \sin x - y_1 \sin x - y\cos x$

$= y_2 \cos x - 2y_1 \sin x - y\cos x$

$= y_2 \cos x - 2y_1 \sin x - y_1$,

and $y_4 = y_3 \cos x - y_2 \sin x - 2(y_2 \sin x + y_1 \cos x) - y_2$

$= y_3 \cos x - 3y_2 \sin x - 2y_1 \cos x - y_2$

and so on.

Now $y(0) = 1$

and $y_1(0) = 1\times 1 = 1$,

$y_2(0) = 1\times 1 - 0 = 1$,

$y_3(0) = 1\times 1 - 0 - 1 = 0$,

$y_4(0) = 0 - 0 - 3 + 0 = -3$ and so on.

Space for Hints

Using Maclaurin's theorem,

(i.e) $y(x) = y(0)+\frac{x}{1!}y_1(0)+\frac{x^2}{2!}y_2(0)+\text{L}$

(i.e) $e^{\sin x} = 1+\frac{x}{1!}(1)+\frac{x^2}{2!}(1)+\frac{x^3}{3!}(0)+\frac{x^4}{4!}(-3)+\text{L}$

(i.e) $e^{\sin x} = 1+x+\frac{x^2}{2}+-\frac{x^4}{8}+\text{L}$

which is the required expansion.

Example 1.2.8 :

Expand $\log\sec x$

Solution : Let $y = f(x) = \log\sec x$

Now $y = f(x) = \log\sec x$.

$\therefore\ y_1 = \frac{1}{\cancel{\sec x}}(\cancel{\sec x}\cdot\tan x) = \tan x$,

and $y_2 = \sec^2 x = 1+\tan^2 x = 1+y_1^2$.

$\therefore\ y_3 = 2y_1y_2$,

$y_4 = 2y_1y_3+2y_2y_2$,

$= 2y_1y_3+2y_2^2$,

$y_5 = 2y_1y_4+2y_2y_3+4y_2y_3$,

$= 2y_1y_4+6y_2y_3$,

$y_6 = 2y_1y_5+2y_2y_4+6y_2y_4+6y_3y_3$,

$= 2y_1y_5+8y_2y_4+6y_3^2$ and so on.

Now $y(0) = \log\sec 0 = \log 1 = 0$

$y_1(0) = \tan 0 = 1$,

$y_2(0) = 1 + 0 = 1$,

$y_3(0) = 0$,

$y_4(0) = 2\ (1) + 0 = 2$,

$y_5(0) = 0 + 0 = 0$,

$y_6(0) = 0+8\times1\times2+0 = 16$ and so on.

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Using Maclaurin's theorem, we have,

$$y(x) = y(0)+\frac{x}{1!}y_1(0)+\frac{x^2}{2!}y_2(0)+\cdots$$

$$\text{(i.e)}\ \log\sec x = \frac{x^2}{2!}(1)+\frac{x^4}{4!}(2)+\frac{x^6}{6!}(16)+\cdots$$

$$\text{(i.e)}\ \log\sec x = \frac{x^2}{2}+\frac{x^4}{12}+\frac{x^6}{45}+\cdots$$

which is the required expansion.

Check your progress

Question :

Prove that $\log(1+e^x) = \log 2+\frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{192}+\cdots$

1.2.3 Leibnitz Formula

This formula is used to find the n^{th} - derivative of the product of two variables in terms of the variables themselves and their successive derivatives.

Statement :

Let u and v be two functions having n^{th} order derivatives. Then $D^n(uv)=u_n v+{}^nC_1 u_{n-1}v_1+{}^nC_2 u_{n-2}v_2+\cdots+uv_n$ where $u_i=\frac{d^i u}{dx^i}$ and $v_j=\frac{d^j u}{dx^j}$.

Proof : We shall prove the theorem using Mathematical induction on n.

If $n=1$ then $D(uv)=u\,D(v)+v\,D(u)$

$$= uv_1+vu_1$$

$$= uv_1+{}^1C_1 vu_1$$

Hence the theorem is true for $n=1$.

Let $n=2$.

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Now $D^2(uv) = \dfrac{d^2}{dx^2}(uv)$

$$= \frac{d}{dx}\left(\frac{d}{dx}(uv)\right)$$

$$= \frac{d}{dx}\left(\frac{du}{dx}\cdot v + u\cdot\frac{dv}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{du}{dx}\cdot v\right) + \frac{d}{dx}\left(u\cdot\frac{dv}{dx}\right)$$

$$= \frac{d^2u}{dx^2}\cdot v + \frac{du}{dx}\cdot\frac{dv}{dx} + \frac{du}{dx}\cdot\frac{dv}{dx} + u\cdot\frac{d^2v}{dx^2}$$

$$= \frac{d^2u}{dx^2}\cdot v + 2\cdot\frac{du}{dx}\cdot\frac{dv}{dx} + u\cdot\frac{d^2v}{dx^2}$$

$$= u_2v + {}^2C_1u_1v_1 + {}^2C_2uv_2.$$

Hence the theorem is true for $n = 2$.

Assume the result is true for $n = k$.

(i.e) $D^k(uv) = u_kv + {}^kC_1u_{k-1}v_1 + \cdots + {}^kC_kuv_k$.

We shall prove the result is true for $n = k+1$.

Now $D^{k+1}(uv) = D(D^k(uv))$

$$= D(u_kv + {}^kC_1u_{k-1}v_1 + {}^kC_2u_{k-2}v_2 + \cdots + uv_k)$$

$$= (u_{k+1}v + u_kv_1) + {}^kC_1(u_kv_1 + u_{k-1}v_2) + \cdots + {}^kC_k(u_1v_k + uv_{k+1})$$

$$= u_{k+1}v + (1 + {}^kC_1)u_kv_1 + ({}^kC_1 + {}^kC_2)u_{k-1}v_2$$

$$+ \cdots + ({}^nC_{r-1} + {}^nC_r)u_{k-r+1}v_1 + \cdots + uv_{k+1}$$

$$= u_{k+1}v + {}^{k+1}C_1u_kv_1 + {}^{k+1}C_2u_{k-1}v_2 + \cdots + uv_{k+1} \quad (\text{since } {}^kC_{r-1} + {}^kC_r = {}^{k+1}C_r)$$

(i.e) the result is true for $n = k+1$.

$\therefore$ by Mathematical Induction, the result is true for any $n \in \mathbb{N}$.

Thus $D^n(uv) = u_nv + {}^nC_1u_{n-1}v_1 + {}^nC_2u_{n-2}v_2 + \cdots + uv_n$.

This proves the theorem.

Example 1.3.1 :

If $y = \sin(\sin^{-1}x)$ prove that

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$$(1-x^2)y_2 - x y_1 + m^2 y = 0 \text{ and also prove that}$$

$$(1-x^2)y_{n+2} - (2n+1)x y_{n+1} + (m^2 - n^2)y = 0.$$

Proof : Given that $y = \sin(\sin^{-1} x)$.

(i.e) $\sin^{-1} y = m\sin^{-1} x$

Differentiate with respect to x, we get,

$$\frac{y_1^2}{1-y^2} = \frac{m^2}{1-x^2}.$$

(i.e) $(1-x^2)y_1^2 = m^2(1-y^2)$.

Again differentiate with respect to x, we get,

$$(1-x^2)2\cdot y_1\cdot y_2 + y_1^2(-2x) = m^2(-2y)y_1.$$

Divide by $2y_1$ on both sides, we get,

$$(1-x^2)\cdot y_2 - 2xy_1 + m^2 y = 0 \quad \text{---} \quad (1.5)$$

This proves the first part of the problem.

Using Leibniz's theorem, taking n^{th} derivative of (1.5), we have,

$$\left[(1-x^2)\cdot y_{n+2} + {}^nC_1(-2x)\cdot y_{n+1} - {}^nC_2\, 2y_n\right]$$

$$-\left[x y_{n+1} + {}^nC_1\cdot 1\cdot y_n\right] + m^2 y_n = 0.$$

(i.e) $(1-x^2)\cdot y_{n+2} - 2nx y_{n+1} - \dfrac{n(n-1)}{\cancel{2}}\cdot\cancel{2}\cdot y_n$

$$-x y_{n+1} - n y_n + m^2 y_n = 0.$$

(i.e) $(1-x^2)\cdot y_{n+2} + (-2nx - x)y_{n+1}$

$$+(-n^2 + \cancel{n} - \cancel{n} + m^2)y_n = 0.$$

(i.e) $(1-x^2)y_{n+2} - (2n+1)x y_{n+1} + (m^2 - n^2)y = 0$.

This proves the problem.

Example 1.3.2 :

If $y = \dfrac{\sinh^{-1} x}{\sqrt{1+x^2}}$, prove that

$$(1+x^2)y_{n+2} + (2n+3)x y_{n+1} + (n+1)^2 y_n = 0.$$

Space for Hints

Proof : Given that $y=\frac{\sinh^{-1}x}{\sqrt{1+x^2}}$.

(i.e) $\sqrt{1+x^2}\,y=\sinh^{-1}x$

(i.e) $\sqrt{1+x^2}\,y=\log\left(x+\sqrt{1+x^2}\right)$

Differentiate with respect to x, we get,

$$\frac{1}{\cancel{2}\sqrt{1+x^2}}\cdot\cancel{2}\,x\,y+\sqrt{1+x^2}\cdot y_1=\frac{1}{x+\sqrt{1+x^2}}\cdot\left(1+\frac{1}{\cancel{2}\sqrt{1+x^2}}\cdot\cancel{2}\,x\right).$$

(i.e) $\frac{x\,y}{\sqrt{1+x^2}}+\sqrt{1+x^2}\cdot y_1=\frac{1}{\cancel{x+\sqrt{1+x^2}}}\cdot\left(\frac{\cancel{x+\sqrt{1+x^2}}}{\sqrt{1+x^2}}\right).$

(i.e) $\frac{x\,y+(1+x^2)\,y_1}{\sqrt{1+x^2}}=\frac{1}{\sqrt{1+x^2}}$

(i.e) $x\,y+(1+x^2)\,y_1=1$

(i.e) $(1+x^2)\,y_1+x\,y=1$

Again differentiate with respect to x, we get,

$$(1+x^2)\,y_2+2x\cdot y_1+x\cdot y_1+y=0$$

(i.e) $(1+x^2)\,y_2+3x\cdot y_1+x\cdot y_1=0$ ------------------------------------- **(1.6)**

Applying Leibniz's theorem and the n^{th} derivative of (1.6) is given by

$(1+x^2)\,y_{n+2}+{}^nC_1\,2x\cdot y_{n+1}+{}^nC_2\,.2.y_n+3x\,y_{n+1}+{}^nC_1\,3\cdot y_n+y_n=0$

(i.e) $(1+x^2)\,y_{n+2}+n\cdot 2x\cdot y_{n+1}+\frac{n(n-1)}{\cancel{2}}\cancel{2}.y_n+3x\,y_{n+1}+3n\,y_n+y_n=0$

(i.e) $(1+x^2)\,y_{n+2}+(n\cdot 2x+3x)y_{n+1}+(n^2-n+3n+1)y_n=0$.

(i.e) $(1+x^2)\,y_{n+2}+(2n+3)\,x\,y_{n+1}+(n^2+2n+1)y_n=0$.

(i.e) $(1+x^2)\,y_{n+2}+(2n+3)\,x\,y_{n+1}+(n+1)^2\,y_n=0$.

This proves the problem.

Space for Hints

Example 1.3.3 :

If $I_n = \frac{d^n}{dx^n}(x^n \log x)$ prove that $I_n = nI_{n-1} + (n-1)!$. Hence deduce that

$$I_n = n!\left\{ \log x + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right\}.$$

Proof : Given that $I_n = \frac{d^n}{dx^n}(x^n \log x)$

Now $I_n = \frac{d^n}{dx^n}(x^n \log x)$

$$= \frac{d^{n-1}}{dx^{n-1}}\left(\frac{d}{dx}(x^n \log x) \right)$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left(x^n \frac{1}{x} + n x^{n-1} \log x \right)$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left(x^{n-1} + n x^{n-1} \log x \right)$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left(x^{n-1} \right) + n\frac{d^{n-1}}{dx^{n-1}}\left(x^{n-1} \log x \right)$$

$$= (n-1)! + nI_{n-1}.$$

Divide by $n!$ on both sides, we get,

$$\frac{I_n}{n!} = \frac{1}{n} + \frac{I_{n-1}}{(n-1)!} \quad \text{--} \quad (1.7)$$

Substituting $n = n-1, n-2, n-3, \cdots, 3, 2$ in (1.7), we have,

$$\frac{I_{n-1}}{(n-1)!} = \frac{1}{n-1} + \frac{I_{n-2}}{(n-2)!}$$

$$\frac{I_{n-2}}{(n-2)!} = \frac{1}{n-2} + \frac{I_{n-3}}{(n-3)!}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\frac{I_3}{3!} = \frac{1}{3} + \frac{I_2}{2!}$$

$$\frac{I_2}{2!} = \frac{1}{2} + \frac{I_1}{1!}$$

Adding above equations and (1.7), we get,

Space for Hints

$$\frac{I_n}{n!}=\frac{1}{n}+\frac{1}{n-1}+\cdots+\frac{1}{3}+\frac{1}{2}+I_1 \quad \text{--} \quad (1.8)$$

Now we shall find I_1.

Now $I_1 = \dfrac{d}{dx}(x\log x)$

$$= x\frac{1}{x}+1\cdot\log x$$

$$= 1+\log x.$$

Thus (1.8) $\Rightarrow \quad \dfrac{I_n}{n!}=\dfrac{1}{n}+\dfrac{1}{n-1}+\cdots+\dfrac{1}{3}+\dfrac{1}{2}+1+\log x.$

(i.e) $I_n = n!\cdot\left(\dfrac{1}{n}+\dfrac{1}{n-1}+\cdots+\dfrac{1}{3}+\dfrac{1}{2}+1+\log x\right).$

This proves the problem.

Example 1.3.4 :

If $y^{1/m}+y^{-1/m}=2x$ prove that $(x^2-1)\,y_{n+2}+(2n+1)\,x\,y_{n+1}+(n^2-m^2)\,y_n=0$

Proof : Given that $y^{1/m}+y^{-1/m}=2x$ ------------------------------------ (1.9)

Put $y^{1/m} = z$.

$\therefore \quad y^{-1/m}=z^{-1}=\dfrac{1}{z}.$

Thus (1.9) $\Rightarrow \quad z+\dfrac{1}{z}=2x$

$\Rightarrow \quad z^2+1=2xz$

$\Rightarrow \quad z^2-2xz+1$

$\Rightarrow \quad z=\dfrac{2x\pm\sqrt{4x^2-4}}{2}$

$\Rightarrow \quad z=\dfrac{\cancel{2}\left(x\pm\sqrt{x^2-1}\right)}{\cancel{2}}$

$\Rightarrow \quad z=x\pm\sqrt{x^2-1}$

(i.e) $y^{1/m} = x\pm\sqrt{x^2-1}$

Space for Hints

$$\Rightarrow y = \left(x \pm \sqrt{x^2-1}\right)^m$$

Differentiate with respect to x, we get,

$$\frac{dy}{dx} = m\left(x \pm \sqrt{x^2-1}\right)^{m-1} \cdot \left(1 \pm \frac{1}{\cancel{2}\sqrt{x^2-1}} \cdot \cancel{2}x\right).$$

$$= m\left(x \pm \sqrt{x^2-1}\right)^{m-1} \cdot \left(\frac{\sqrt{x^2-1} \pm x}{\sqrt{x^2-1}}\right).$$

$$= m\frac{\left(x \pm \sqrt{x^2-1}\right)^m}{\sqrt{x^2-1}}.$$

(i.e) $y_1 = \dfrac{my}{\sqrt{x^2-1}}$

Squaring on both sides, we get, $y_1^2 = \dfrac{m^2 y^2}{x^2-1}$.

(i.e) $(x^2-1)\,y_1^2 = m^2 y^2$

Again differentiate with respect to x, we get,

$(x^2-1)\cdot 2 \cdot y_1 \cdot y_2 + 2x \cdot y_1^2 = 2m^2 y\, y_1$.

Divide by $2y_1$, we get,

$(x^2-1)\cdot y_2 + x \cdot y_1 = m^2 y$ -- (1.10)

Using Leibniz's theorem, differentiate (1.10) n times, we get,

$(x^2-1)y_{n+2} + {}^nC_1\, 2xy_{n+1} + {}^nC_2 \cdot 2y_n + x \cdot y_{n+1} + {}^nC \cdot 1 \cdot y_n = m^2 y_n$

(i.e) $(x^2-1)\,y_{n+2} + n \cdot 2x \cdot y_{n+1} + \dfrac{n(n-1)}{\cancel{2}}\cancel{2}.y_n + x\,y_{n+1} + n\,y_n - m^2 y_n = 0$

(i.e) $(x^2-1)\,y_{n+2} + (2n+1)\,x \cdot y_{n+1} + (n^2 - \cancel{n} + \cancel{n} - m^2)\cdot y_n = 0$

(i.e) $(x^2-1)\,y_{n+2} + (2n+1)\,x \cdot y_{n+1} + (n^2 - m^2)\cdot y_n = 0$.

This proves the problem.

Example 1.3.5 :

If $y = \left(x + \sqrt{1+x^2}\right)^m$, then prove that

$(1+x^2)\,y_{n+2} + (2n+1)\,x \cdot y_{n+1} + (n^2 - m^2)\cdot y_n = 0$.

Space for Hints

Proof : Given that $y=\left(x+\sqrt{1+x^2}\right)^m$

Differentiate with respect to x, we get,

$$y_1 = m\left(x+\sqrt{1+x^2}\right)^{m-1}\cdot\left(1+\frac{1}{\cancel{2}\sqrt{1+x^2}}\cdot\cancel{2}x\right)$$

$$= m\left(x+\sqrt{1+x^2}\right)^{m-1}\cdot\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right).$$

$$= m\frac{\left(x+\sqrt{1+x^2}\right)^m}{\sqrt{1+x^2}}.$$

(i.e) $y_1 = \dfrac{my}{\sqrt{1+x^2}}$

Squaring on both sides, we get, $y_1^2 = \dfrac{m^2y^2}{1+x^2}$.

(i.e) $(1+x^2)y_1^2 = m^2y^2$ -- (1.11)

Using Leibniz's theorem, differentiate (1.11) n times, we get,

$(1+x^2)\cdot 2\cdot y_1\cdot y_2 + 2x\cdot y_1^2 = 2m^2yy_1$.

Divide by $2y_1$, we get,

$(1+x^2)\cdot y_2 + x\cdot y_1 = m^2y$

$(1+x^2)y_{n+2} + {}^nC_1\, 2xy_{n+1} + {}^nC_2\cdot 2y_n + x\cdot y_{n+1} + {}^nC\cdot 1\cdot y_n = m^2y_n$

(i.e) $(1+x^2)y_{n+2} + (2n+1)x\cdot y_{n+1} + (n^2-\cancel{n}+\cancel{n}-m^2)\cdot y_n = 0$

(i.e) $(1+x^2)y_{n+2} + (2n+1)x\cdot y_{n+1} + (n^2-m^2)\cdot y_n = 0$.

This proves the problem.

Check your progress

Questions :

(1) Prove that $\dfrac{d^n}{dx^n}\left(\dfrac{\log x}{x}\right) = (-1)^n\dfrac{n!}{x^{n+1}}\left[\log x - 1 - \dfrac{1}{2} - \dfrac{1}{3} - \text{L} - \dfrac{1}{n}\right]$.

(2) If $y=\sin^{-1}x$ prove that $(1-x^2)y_2 - xy_1 = 0$ and

$(1-x^2)y_{n+2} - (2n+1)x\cdot y_{n+1} - xy_n = 0$.

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1.4 Maximum and Minimum of Functions of two variables.

In this section we shall discuss the maximum or minimum value of a function $f(x,y)$.

Definition : A function $f(x,y)$ has a maximum value at (a,b) if $f(a,b) > f(x,y)$ for every (x,y) in the neighborhood of (a,b).

(i.e) $f(a,b) > f(a+h,b+k)$ for sufficiently small h, k.

Definition : A function $f(x,y)$ has a minimum value at (a,b) if $f(a,b) < f(x,y)$ for every (x,y) in the neighborhood of (a,b).

(i.e) $f(a,b) < f(a+h,b+k)$ for sufficiently small h, k.

Procedure for finding maximum or minimum of $f(x,y)$

Let $f(x,y)$ be a given function.

Step 1 : Find $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial y^2}, \dfrac{\partial^2 f}{\partial x \partial y}$.

Step 2 : Solve the equations $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$.

Let $x = a$, $y = b$ be a solution of the above equations.

Step 3 : $f(x,y)$ is said to attain a maximum at (a,b) if

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)} \cdot \left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)} > \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(a,b)},$$

$$\left(\frac{\partial f}{\partial x}\right)_{(a,b)} < 0 \text{ and } \left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)} < 0.$$

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Step 4 : $f(x,y)$ is said to attain a maximum at (a,b) if

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)} \cdot \left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)} > \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2_{(a,b)},$$

$$\left(\frac{\partial f}{\partial x}\right)_{(a,b)} > 0 \text{ and } \left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)} > 0.$$

Note :

We denote $\frac{\partial^2 f}{\partial x^2}$ as r, $\frac{\partial^2 f}{\partial x \partial y}$ as s and $\frac{\partial^2 f}{\partial y^2}$ as t.

(i.e) $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$ and $t = \frac{\partial^2 f}{\partial y^2}$.

Example 1.4.1 :

Find the minimum value of $x^2 - xy + y^2 + 3x - 2y + 1$.

Solution : Let $f(x,y) = x^2 - xy + y^2 + 3x - 2y + 1$.

Differentiate partially with respect to x, y, we get,

$$\frac{\partial f}{\partial x} = 2x - y + 3,$$

$$\frac{\partial f}{\partial y} = -x + 2y - 2 \;,$$

$$\frac{\partial^2 f}{\partial x^2} = 2, \; \frac{\partial^2 f}{\partial y^2} = 2, \; \frac{\partial^2 f}{\partial x \partial y} = -1$$

If $\frac{\partial f}{\partial x} = 0$ then $2x - y + 3 = 0$ -- (1.12) and

If $\frac{\partial f}{\partial y} = 0$ then $-x + 2y - 2 = 0$ -- (1.13)

Solving (1.12) and (1.13), we get, $x = -\frac{4}{3}$, $y = \frac{1}{3}$.

At $\left(-\frac{4}{3}, \frac{1}{3}\right)$,

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$$rt - s^2 = \left(\frac{\partial^2 f}{\partial x^2}\right)\cdot\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$= 2\times 2 - 1$$

$$= 3 > 0$$

Again at $\left(-\frac{4}{3}, \frac{1}{3}\right)$, $\frac{\partial^2 f}{\partial x^2} = 2 > 0$ and $\frac{\partial^2 f}{\partial y^2} = 2 > 0$.

Thus $f(x,y)$ attains its minimum at $\left(-\frac{4}{3}, \frac{1}{3}\right)$ and the minimum value is

$$f\left(-\frac{4}{3}, \frac{1}{3}\right) = \left(-\frac{4}{3}\right)^2 - \left(-\frac{4}{3}\right)\cdot\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 + 3\left(-\frac{4}{3}\right) - 2\left(\frac{1}{3}\right) + 1.$$

$$= -\frac{4}{3}.$$

Example 1.4.2 :

Examine for extreme values of the function $x^2y^2 - 5x^2 - 8xy - 5y^2$.

Solution : Let $f(x,y) = x^2y^2 - 5x^2 - 8xy - 5y^2$.

Differentiate partially with respect to *x*, *y*, we have,

$$\therefore \frac{\partial f}{\partial x} = 2xy^2 - 10x - 8y,$$

$$\frac{\partial f}{\partial y} = 2x^2y - 8x - 10y.$$

Again $r = \frac{\partial^2 f}{\partial x^2} = 2y^2 - 10$,

$t = \frac{\partial^2 f}{\partial y^2} = 2x^2 - 10$ and $s = 4xy - 8$.

For maximum or minimum $f(x,y)$, put $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

(i.e) $2xy^2 - 10x - 8y = 0$ and

$2x^2y - 8x - 10y = 0$

(i.e) $xy^2 - 5x - 4y = 0$ -- (1.14) and

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$$x^2y-4x-5y=0 \quad \text{--------------------------------} \quad (1.15)$$

Now (1.15) – (1.14) $\Rightarrow$ $x^2y-4x-5y-xy^2+5x+4y=0$

$\Rightarrow$ $xy(x-y)+x-y=0$

$\Rightarrow$ $(x-y)(xy+1)=0$

$\Rightarrow$ $(x-y)=0$ or $xy+1=0$

$\Rightarrow$ $x=y$ or $xy=-1$

$\Rightarrow$ $x=y$ or $y=-\dfrac{1}{x}$.

When $x=y$ then (1.14) $\Rightarrow$ $x^3-5x-4x=0$

$\Rightarrow$ $x^3-9x=0$

$\Rightarrow$ $x(x^2-9)=0$

$\Rightarrow$ $x=0$ or $x^2-9=0$

$\Rightarrow$ $x=0$ or $x^2=9$

$\Rightarrow$ $x=0$ or $x=\pm 3$

$\therefore$ the points are $(0,0)$, $(3,3)$, $(-3,-3)$.

Again, when $y=-\dfrac{1}{x}$ then (1.14) $\Rightarrow$ $x\cdot\dfrac{1}{x^2}-5x+\dfrac{4}{x}=0$

$\Rightarrow$ $\dfrac{5}{x}-5x=0$

$\Rightarrow$ $5-5x^2=0$

$\Rightarrow$ $x^2=1$

$\Rightarrow$ $x=\pm 1$

$\therefore$ the points are $(1,-1)$, $(-1,1)$.

Hence the critical points are $(0,0)$, $(3,3)$, $(-3,-3)$, $(1,-1)$, $(-1,1)$.

Now $rt-s^2 = 4(x^2-5)(y^2-5)-16(xy-2)^2$

At $(0,0)$, $rt-s^2 = 100-64 = 36 > 0$ and $r = -10 < 0$.

$\therefore$ $f(x,y)$ is maximum at $(0,0)$ and the maximum value is $f(0,0) = 0$.

At $(3,3)$, $rt-s^2 = 4\,(4)\,(4) - 16\,(49) = -720 < 0$.

$\therefore$ there is no extremum at $(3,3)$.

At $(-3,-3)$, $rt-s^2 = 4\,(4)\,(4) - 16\,(49) = -720 < 0$.

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Here also no extremum at $(3,3)$.

At $(1,-1)$, $rt-s^2 = 4(-4)(-4)-16(9) = -80 < 0$.

$\therefore$ there is no extremum at $(1,-1)$.

At $(-1,1)$, $rt-s^2 = 4(-4)(-4)-16(9) = -80 < 0$.

$\therefore$ there is no extremum at $(-1,1)$.

Example 1.4.3 :

Find the maximum or minimum value of $x^3+3xy^2-15x^2-15y^2+72x$.

Solution : Let $f(x,y) = x^3+3xy^2-15x^2-15y^2+72x$.

Differentiate partially with respect to x, y, we have,

$$\frac{\partial f}{\partial x} = 3x^2+3y^2-30x+72,$$

$$\frac{\partial f}{\partial y} = 6xy-30y.$$

Again $r = \dfrac{\partial^2 f}{\partial x^2} = 6x-30$,

$t = \dfrac{\partial^2 f}{\partial y^2} = 6x-30$ and $s = 6y$.

For maximum or minimum $f(x,y)$, put $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$.

(i.e) $3x^2+3y^2-30x+72=0$ and

$6xy-30y=0$.

(i.e) $x^2+y^2-10x+24=0$ -- (1.16) and

$xy-5y=0$ -- (1.17)

From (1.17), $y(x-5)=0 \Rightarrow y=0$ or $x=5$.

When $y=0$ then (1.16) $\Rightarrow x^2-10x+24=0$.

$\Rightarrow x=4$ or $x=6$.

(i.e) the points are $(4,0)$, $(6,0)$.

When $x=5$ then (1.16) $\Rightarrow 25+y^2-50+24=0$.

$\Rightarrow y^2=1$.

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$\Rightarrow y = \pm 1$.

(i.e) the points are $(5,1)$, $(5,-1)$.

Hence the critical points are $(4,0)$, $(6,0)$, $(5,1)$, $(5,-1)$.

At $(4,0)$, $rt - s^2 = 36\,(1) - 0 = 36 > 0$ and

$r = 24 - 30 = -6 < 0$.

$\therefore f(x,y)$ is maximum at $(4,0)$ and the maximum value is

$f(4,0) = 64 - 240 + 288 = 112$.

At $(6,0)$, $rt - s^2 = 36\,(1) - 0 = 36 > 0$ and

$r = 36 - 30 = 6 > 0$.

$\therefore f(x,y)$ is minimum at $(6,0)$ and the minimum value is

$f(6,0) = 216 - 540 + 432 = 108$.

At $(5,1)$, $rt - s^2 = 0 - 36 = -36 < 0$.

There is no extremum at $(5,1)$.

At $(5,-1)$, $rt - s^2 = 0 - 36 = -36 < 0$.

There is no extremum at $(5,-1)$.

Example 1.4.4 :

Discuss the maxima and minima of $x^3 y^2 (1 - x - y)$.

Solution : Let $f(x,y) = x^3 y^2 (1 - x - y)$.

(i.e) $f(x,y) = x^3 y^2 - x^4 y^2 - x^3 y^3$.

Differentiate partially with respect to *x*, *y*, we have,

$$\frac{\partial f}{\partial x} = 3x^2 y^2 - 4x^3 y^2 - 3x^2 y^3,$$

$$\frac{\partial f}{\partial y} = 2x^3 y - 2x^4 y - 3x^3 y^2.$$

Again $r = \dfrac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2 y^2 - 6xy^3$,

$$t = \frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3 y$$

and $s = 6x^2 y - 8x^3 y - 9x^2 y^2$.

For maximum or minimum $f(x,y)$, put $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$.

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Now $\frac{\partial f}{\partial x}=0 \Rightarrow 3x^2y^2-4x^3y^2-3x^2y^3=0$

$\Rightarrow x^2y^2(3-4x-3y)=0$

$\Rightarrow x=0$ or $y=0$ or $3-4x-3y=0$ ------------------------------------ (1.18)

Again $\frac{\partial f}{\partial y}=0 \Rightarrow 2x^3y-2x^4y-3x^3y^2=0$

$\Rightarrow x^3y(2-2x-3y)=0$

$\Rightarrow x=0$ or $y=0$ or $2-2x-3y=0$ ------------------------------------ (1.19)

Solving (1.18) and (1.19), we get, the critical points $(0,0)$ and $\left(\frac{1}{2},\frac{1}{3}\right)$.

Now $rt-s^2$

$= (6xy^2-12x^2y^2-6xy^3)\cdot(2x^3-2x^4-6x^3y)-(6x^2y-8x^3y-9x^2y^2)^2$

$= 12x^4y^2(1-2x-y)(1-x-3y)-(6-8x-9y)^2$.

At $(0,0)$, $rt-s^2=0$.

(i.e) $f(x,y)$ cannot be say about maximum or minimum at $(0,0)$.

At $\left(\frac{1}{2},\frac{1}{3}\right)$, $rt-s^2 = 12\cdot\frac{1}{16}\cdot\frac{1}{9}\left(-\frac{1}{3}\right)\left(--\frac{1}{2}\right)-\frac{1}{16}\cdot\frac{1}{9}\cdot 1$

$$= \frac{1}{144} > 0$$

and $r = 6\cdot\frac{1}{2}\cdot\frac{1}{9}-12\cdot\frac{1}{4}\cdot\frac{1}{9}-6\cdot\frac{1}{2}\cdot\frac{1}{27}$

$$= -\frac{1}{9} < 0.$$

$\therefore f(x,y)$ has a maximum value at $\left(\frac{1}{2},\frac{1}{3}\right)$ and maximum value is $f\left(\frac{1}{2},\frac{1}{3}\right)$

$= \frac{1}{8}\cdot\frac{1}{9}\cdot\left(1-\frac{1}{2}-\frac{1}{3}\right) = \frac{1}{432}$.

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Summary

In this unit we have leaned how to find the derivatives of higher order, that is the successive differentiation, how to expand a given function, how to apply Leibnitz Theorem, and the method of finding maximum or minimum of a function containing two variables.

Further Reading

You can also refer the following books for further reading.

(1) Calculus by Arumugam and Isaac

(2) Differential Calculus by Shanti Narayanan

UNIT II

Space for
Hints

SUBTANGENT AND SUBNORMAL

Space for Hints

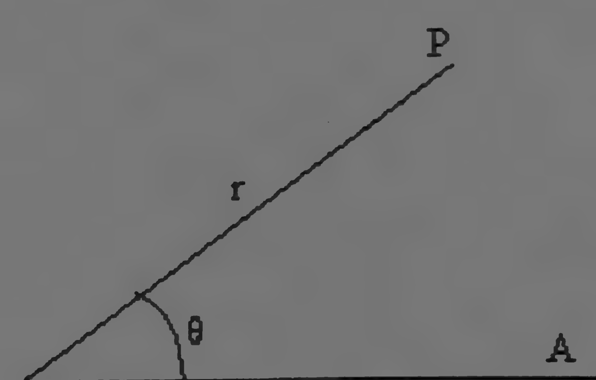
Objectives :

In this unit we are going to study subtangent and subnormal, angle between the radius vector and the tangent, slope of the tangent, angle of intersection of two curves, polar subtangent and subnormal and length of arc.

After completing this unit, students may able to know

- Subtangent and subnormal
- Angle between the radius vector and the tangent
- Slope of the tangent
- Angle of intersection of two curves
- Polar subtangent and subnormal
- Length of arc

Introduction



The position of a point P on a plane can be represented in two ways, first one is Cartesian coordinates form and another one is polar coordinates form. Here we shall use polar coordinates. The method of representing a point P using polar coordinates is given below :

Consider a fixed point O called as pole and a fixed line OA called as initial line.

A point P can be identified as (r,θ) where $OP = r$ and $A\hat{O}P = \theta$.

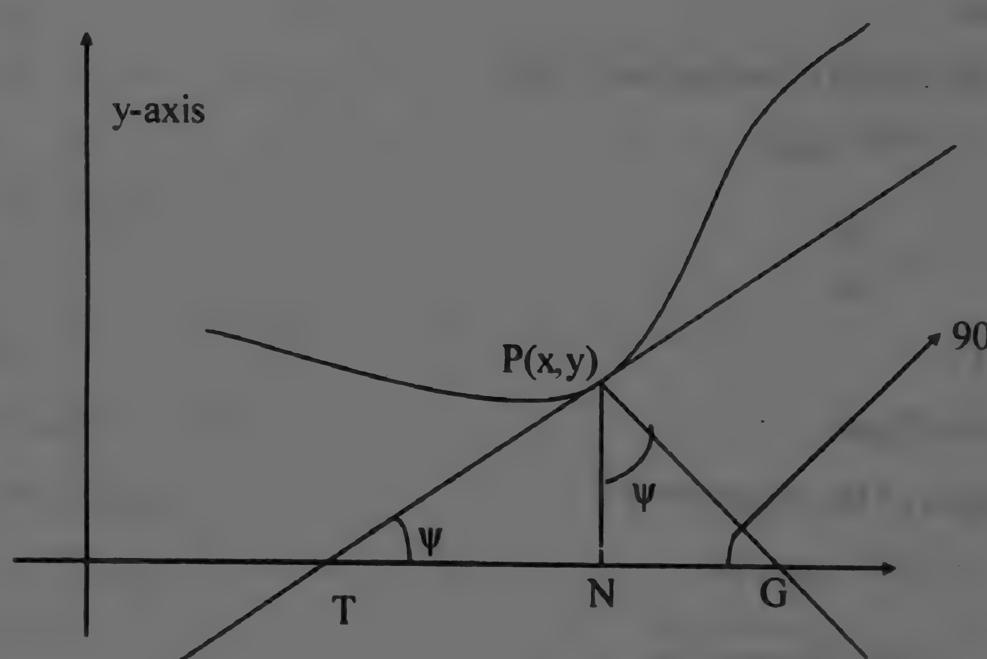
Here r is called radius vector and θ is called vectorial angle and it is measured in the anti-clockwise direction.

Note :

The relationship between the Cartesian coordinates and polar coordinates are give below :

$x = r\cos\theta$, $y = r\sin\theta$.

2.1. Subtangent and subnormal



Consider the above figure.

Let $y = f(x)$ be a given function.

Let $P(x,y)$ be appoint on the curve $y = f(x)$

Draw tangent and normal at P and let they meet the x–axis at T and G respectively.

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Draw PN perpendicular to x – axis.

Let the tangent makes an angle ψ with the x – axis.

Then $\tan\psi = \frac{dy}{dx}$ and $\cot\psi = \frac{1}{\tan\psi} = \frac{dy}{dx}$.

In the figure, then length TN is called subtangent and NG is called subnormal.

Definition : (Subtangent)

The subtangent for any point on the curve is the projection of the tangent on x – axis.

$\therefore$ The length of subtangent = TN

$$= PN\cot\psi$$

$$= y\cdot\frac{dx}{dy}$$

Definition : (Subnormal)

The subnormal for any point on the curve is the projection of the normal on x – axis.

Thus the length of subnormal = NG

$$= PN\cdot\tan\psi$$

$$= y\cdot\frac{dy}{dx}$$

Note 1 :

From the figure,

the length of the tangent = PT

$$= PN\cdot\operatorname{cosec}\psi$$

$$= PN\cdot\sqrt{1+\cot^2\psi}$$

$$= y\cdot\sqrt{1+\left(\frac{dx}{dy}\right)^2}$$

Note 2 : Length of normal = PG

$$= PN \cdot \sec\psi$$

$$= PN \cdot \sqrt{1+\tan^2\psi}$$

$$= PN \cdot \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Space for Hints

Example 2.1.1 :

Prove that the length of the subtangent is constant for the curve $y = a^x$.

Solution : Given that $y = a^x$.

$$\therefore \quad \frac{dy}{dx} = a^x \cdot \log a$$

(i.e) $\dfrac{dy}{dx} = y \cdot \log a$.

Thus the length of the subtangent

$$= y \cdot \frac{dx}{dy}$$

$$= \cancel{y} \cdot \frac{1}{\cancel{y} \cdot \log a}$$

$$= \frac{1}{\log a}$$

which is a constant.

This proves the problem.

Example 2.1.2 :

Show that the subtangent at any point of the curve $x^m y^n = a^{m+n}$ varies as the abscissa of the point.

Proof : Given that $x^m y^n = a^{m+n}$.

Taking log on both sides, we have,

$$\log\left(x^m y^n\right) = \log\left(a^{m+n}\right)$$

Space for Hints

(i.e) $\log x^{m} + \log y^{n} = (m+n)\log a$

(i.e) $m\log x + n\log y = (m+n)\log a$

Differentiate with respect to x, we get,

$$m \cdot \frac{1}{x} + n \cdot \frac{1}{y} \cdot \frac{dy}{dx} = 0$$

(i.e) $\frac{n}{y} \cdot \frac{dy}{dx} = -\frac{m}{x}$

$$\therefore \ \frac{dy}{dx} = -\frac{m\,y}{n\,x}.$$

Thus subtangent = $y \cdot \frac{dx}{dy}$

$$= \cancel{y} \cdot \left(-\frac{n\,x}{m\,\cancel{y}} \right)$$

$$= -\frac{n\,x}{m} \propto \text{x}$$

$\therefore$ subtangent varies as the abscissa.

This proves the problem.

Example 2.1.3 :

Find the length of the normal at any point P of the rectangular hyperbola $x^2 - y^2 = a^2$ and show that it is equal to the distance of P from the origin.

Solution : Given that $x^2 - y^2 = a^2$

Differentiate with respect to x, we get,

$$2x - 2y\frac{dy}{dx} = 0.$$

$$\therefore \ \frac{dy}{dx} = \frac{x}{y}.$$

Thus the length of normal = $y \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

$$= y\cdot\sqrt{1+\frac{x^2}{y^2}}$$

$$= y\cdot\sqrt{\frac{y^2+x^2}{y^2}}$$

$$= \cancel{y}\cdot\frac{\sqrt{y^2+x^2}}{\cancel{y}}$$

$$= \sqrt{x^2+y^2}$$

which is the distance of the point $P(x,y)$ form the origin.

This proves the problem.

Example 2.1.4 :

Prove that the subnormal at any point of the curve $y^2x^2=a^2(x^2-a^2)$ varies inversely as the cube of its abscissa.

Proof : Given that $y^2x^2=a^2(x^2-a^2)$

Taking log on both sides, we get,

Differentiate with respect to x, we get,

$$2\cdot\frac{1}{y}\cdot\frac{dy}{dx}+2\frac{1}{x}=0+\frac{1}{x^2-a^2}\cdot 2x$$

(i.e) $\cancel{2}\cdot\frac{1}{y}\cdot\frac{dy}{dx}=\frac{\cancel{2}x}{x^2-a^2}\cdot-\cancel{2}\frac{1}{x}$

(i.e) $\frac{1}{y}\cdot\frac{dy}{dx}=\frac{a^2}{x(x^2-a^2)}$

(i.e) $\frac{dy}{dx}=\frac{a^2y}{x(x^2-a^2)}$

(i.e) $\frac{dy}{dx}=\frac{a^2y}{x\cdot\frac{y^2x^2}{a^2}}$

Space for Hints

Space for Hints

(i.e) $\dfrac{dy}{dx} = \dfrac{a^4}{x^3 y}$.

$\therefore$ the length of the subnormal

$$= y \cdot \frac{dy}{dx}$$

$$= \cancel{y} \cdot \frac{a^4}{x^3 \cancel{y}}$$

$$= \frac{a^4}{x^3}$$

$$\propto \frac{1}{x^3}$$

(i.e) the subnormal varies inversely proportional to the cube of abscissa of point.

This proves the problem.

Example 2.1.5 :

Find the length of the subtangent and subnormal at $\phi = \dfrac{\pi}{4}$ when the equation of the ellipse is given by $x = a\cos\phi$, $y = b\sin\phi$. Also find the length of the tangent and normal.

Solution : Given that $x = a\cos\phi$, $y = b\sin\phi$.

$$\therefore \frac{dx}{d\phi} = -a\sin\phi \text{ and } \frac{dy}{d\phi} = b\cos\phi.$$

Thus $\dfrac{dy}{dx} = \dfrac{dy/d\phi}{dx/d\phi}$

$$= \frac{b\cos\phi}{-a\sin\phi}$$

$$= -\frac{b}{a}\cot\phi$$

At $\phi = \dfrac{\pi}{4}$, $y = b\sin\dfrac{\pi}{4} = \dfrac{b}{\sqrt{2}}$

and $\dfrac{dy}{dx} = -\dfrac{b}{a}\cot\dfrac{\pi}{4} = -\dfrac{b}{a}$.

Now length of subtangent = $y\cdot\dfrac{dx}{dy}$

$$= \left|\frac{b}{\sqrt{2}}\cdot\left(-\frac{a}{b}\right)\right|$$

$$= \left|-\frac{a}{\sqrt{2}}\right|$$

$$= \frac{a}{\sqrt{2}},$$

and the length of subnormal = $y\cdot\dfrac{dy}{dx}$

$$= \left|\frac{b}{\sqrt{2}}\cdot\left(-\frac{b}{a}\right)\right|$$

$$= \left|-\frac{b^2}{a\sqrt{2}}\right|$$

$$= \frac{b^2}{a\sqrt{2}},$$

Length of tangent = $y\cdot\sqrt{1+\left(\dfrac{dx}{dy}\right)^2}$

$$= \frac{b}{\sqrt{2}}\cdot\sqrt{1+\frac{a^2}{b^2}}$$

$$= \frac{b}{\sqrt{2}}\cdot\sqrt{\frac{b^2+a^2}{b^2}}$$

$$= \frac{\cancel{b}}{\sqrt{2}}\cdot\frac{\sqrt{a^2+b^2}}{\cancel{b}}$$

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$$= \sqrt{\frac{a^2+b^2}{2}}$$

Length of normal = $y \cdot \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$= \frac{b}{\sqrt{2}} \cdot \sqrt{1+\frac{b^2}{a^2}}$$

$$= \frac{b}{\sqrt{2}} \cdot \sqrt{\frac{a^2+b^2}{a^2}}$$

$$= \frac{b}{a} \cdot \sqrt{\frac{a^2+b^2}{2}}$$

Example 2.1.6 :

Find the length of tangent, normal, subtangent and subnormal to $x = a(t+\sin t)$, $y = a(1-\cos t)$ at t.

Solution : Given that $x = a(t+\sin t)$, $y = a(1-\cos t)$

Now $\frac{dx}{dt} = a(1+\cos t)$ and $\frac{dy}{dt} = a\sin t$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dy/dt}$$

$$= \frac{a\sin t}{a(1+\cos t)}$$

$$= \frac{\cancel{a}\,\cancel{2}\sin(t/2)\cdot\cos(t/2)}{\cancel{a}\,\cancel{2}\cos^2(t/2)}$$

$$= \tan(t/2)$$

$\therefore$ length of the tangent = $y.\sqrt{1+\left(\frac{dx}{dy}\right)^2}$

$$= a(1-\cos t).\sqrt{1+\cot^2(t/2)}$$

$$= a\,2\sin^2(t/2)\cdot \cos ec(t/2) = 2a\sin(t/2)$$

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Length of normal = $y \cdot \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$= a(1-\cos t).\sqrt{1+\tan^2(t/2)}$

$= a2\sin^2(t/2)\cdot sec(t/2)$

$= a2\sin(t/2)\cdot\tan(t/2)$

Now length of subtangent = $y \cdot \frac{dx}{dy}$

$= a(1-\cos t).\cot(t/2)$

$= a2\sin^2(t/2).\frac{\cos(t/2)}{\sin(t/2)}$

$= 2a\sin(t/2)$

Length of subnormal = $y \cdot \frac{dy}{dx}$

$= a(1-\cos t).\tan(t/2)$

Check your progress :

Question :

Find the lengths of tangent, normal, subtangent and subnormal to $x = a\cos^3\theta$, $y = \sin^3\theta$ at θ.

(**Answer :** Length of tangent = $a\sin^2\theta$,

Length of normal = $a\sin^2\theta\tan\theta$,

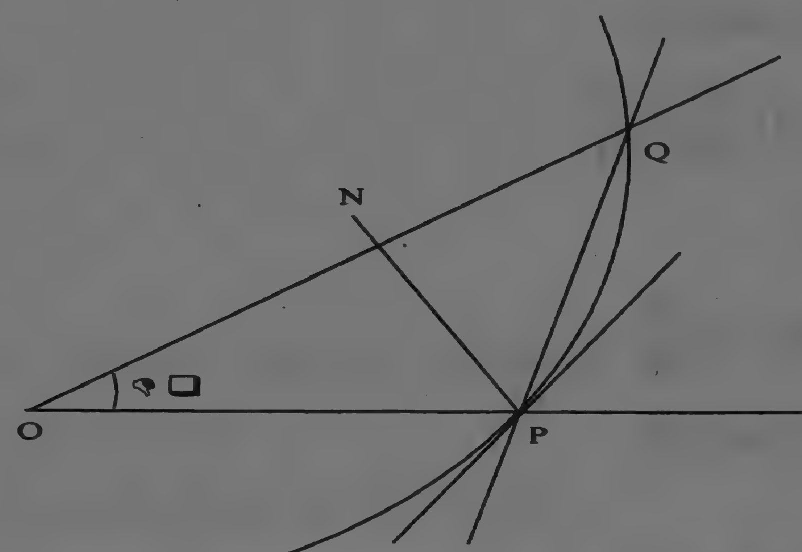
Length of subtangent = $a\sin^2\theta\cos\theta$,

Length of subnormal = $a\sin^3\theta\tan\theta$)

Space for Hints

2.2. Angle between the radius vector and the tangent

Now we shall find the angle between the radius vector and the tangent.



Consider two points P and Q on a curve.

Let (r,θ) be the polar coordinates of P and that of Q be $(r+\Delta r,\theta+\Delta\theta)$.

Join P and Q and draw PN perpendicular to OQ.

Then $PN = OP\sin P\hat{O}N$

$= r\sin\Delta\theta$

and $QN = OQ - ON$

$= r+\Delta r - r\cos\Delta\theta$

$= \Delta r + r(1-\cos\Delta\theta)$

$= \Delta r + 2r\sin^2\left(\dfrac{\Delta\theta}{2}\right).$

Let φ be the OP and the tangent at P.

If $\Delta\theta \to 0$, then we have the following,

(i) the point Q will approach P,

(ii) the secant PQ will become the tangent PT,

(iii) the angle $P\hat{Q}N$ will approach φ.

Now $\tan P\hat{Q}O = \dfrac{PN}{QN}$

$$= \frac{r\sin\Delta\theta}{\Delta r + 2r\sin^2\left(\frac{\Delta\theta}{2}\right)}$$

$$= r\cdot\frac{\dfrac{\sin\Delta\theta}{\Delta\theta}}{\dfrac{\Delta r}{\Delta\theta} + \dfrac{r\sin\left(\frac{\Delta\theta}{2}\right)}{\frac{\Delta\theta}{2}}\cdot\sin\left(\frac{\Delta\theta}{2}\right)}$$

We know that

(1) $\lim\limits_{\Delta\theta\to 0}\dfrac{\sin\Delta\theta}{\Delta\theta} = 1$,

(2) $\lim\limits_{\Delta\theta\to 0}\dfrac{\sin\Delta\theta}{2} = 0$,

(3) $\lim\limits_{\frac{\Delta\theta}{2}\to 0}\dfrac{\sin\left(\frac{\Delta\theta}{2}\right)}{\frac{\Delta\theta}{2}} = 1$, and

(4) $\lim\limits_{\Delta\theta\to 0}\dfrac{\Delta r}{\Delta\theta} = \dfrac{dr}{d\theta}$.

Therefore $\tan\varphi = \lim\limits_{\Delta\theta\to 0}\tan P\hat{Q}O$

$$= r\cdot\frac{1}{\dfrac{dr}{d\theta} + r\cdot 1\cdot 0}$$

$$= r\cdot\frac{d\theta}{dr}.$$

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Hence the angle between the radius vector and the tangent can be obtained from $\tan\varphi = r \cdot \dfrac{d\theta}{dr}$.

(i.e) $\tan\varphi = r \cdot \dfrac{d\theta}{dr}$

Example 2.2.1 :

Find the angle at which the radius vector cuts the curve $\dfrac{l}{e} = 1 + e\cos\theta$.

Solution : Let ϕ be the angle between the radius vector and the tangent at the point at which the radius vector meets the curve.

$\therefore \quad \tan\phi = r \cdot \dfrac{d\theta}{dr}$.

Given that $\dfrac{l}{e} = 1 + e\cos\theta$.

Differentiate with respect to θ, we get,

$$-\frac{l}{r^2} \cdot \frac{dr}{d\theta} = -e\sin\theta$$

$$\Rightarrow \quad \frac{dr}{d\theta} = \frac{r^2 e}{l} \cdot \sin\theta$$

Thus $\tan\phi = r \cdot \dfrac{d\theta}{dr}$

$$= r \cdot \frac{l}{e \cdot r^2 \cdot \sin\theta}$$

$$= \frac{l}{e \cdot r \cdot \sin\theta}$$

$$= \frac{1 + e\cos\theta}{e \cdot r \cdot \sin\theta}$$

$$\therefore \quad \phi = \tan^{-1}\left(\frac{1 + e\cos\theta}{e \cdot r \cdot \sin\theta}\right).$$

Example 2.2.2 :

Find ϕ in terms of θ for the curve $r^2 = a^2 \cos 2\theta$.

Solution : Given that $r^2 = a^2 \cos 2\theta$

Now differentiate with respect to θ, we get,

$$\cancel{2} r \frac{dr}{d\theta} = -\cancel{2} a^2 \sin 2\theta$$

(i.e) $$\frac{dr}{d\theta} = \frac{-2a^2 \sin 2\theta}{r}.$$

Thus $$\tan\phi = r \cdot \frac{d\theta}{dr}$$

$$= r\left(\frac{-r}{a^2 \sin 2\theta}\right)$$

$$= \frac{-r^2}{a^2 \sin 2\theta}$$

$$= \frac{-\cancel{a^2} \cos 2\theta}{\cancel{a^2} \sin 2\theta}$$

$$= -\cot 2\theta$$

$$= \cot\left(\frac{\pi}{2} + 2\theta\right)$$

$$\boxed{\therefore \ \phi = \frac{\pi}{2} + 2\theta}$$

Check your progress :

Question 3 :

Find ϕ in terms of θ for the curve $\frac{2a}{r} = 1 - \cos\theta$.

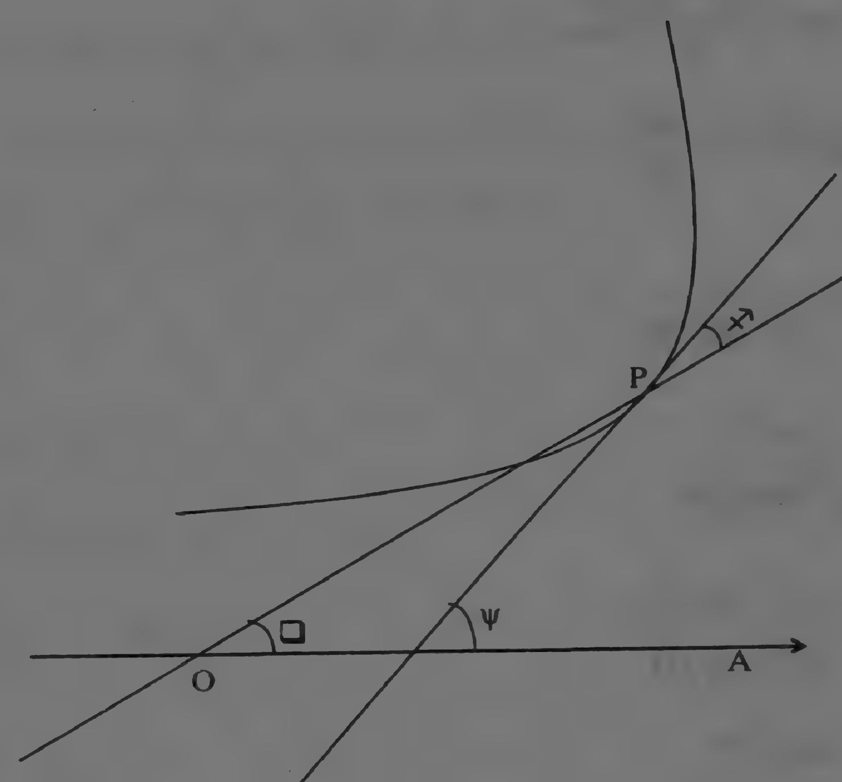
Space for Hints

Space for Hints

2.3. Slope of the tangent in polar coordinates

Book work : Find the Slope of the tangent in polar coordinates

Solution :



Let O be the pole and OA be the initial line.

Let $P(r, \theta)$ be a point on the curve.

Let ψ be the angle made by the tangent at P with the initial line.

Now $\psi = \phi + \theta$

$$\therefore \quad \tan\psi = \tan(\phi + \theta)$$

$$= \frac{\tan\phi + \tan\theta}{1 - \tan\phi \cdot \tan\theta}.$$

If the equation of the curve is known then $\tan\phi$ can be calculated (using previous section) and hence we can able to find $\tan\psi$.

Example 2.3.1 :

Space for Hints

Find the slope of $r=(1-\cos\theta)$ at $\theta=\frac{\pi}{2}$.

Proof : Given that $r=(1-\cos\theta)$

Differentiate with respect to θ, we get, $\frac{dr}{d\theta}=a\sin\theta$.

$$\therefore\quad \tan\phi = r\cdot\frac{dr}{d\theta}$$

$$= \frac{\cancel{a}(1-\cos\theta)}{\cancel{a}\sin\theta}$$

At $\theta=\frac{\pi}{2}$, then $\tan\phi = \frac{1}{1} = 1$

$$\Rightarrow \phi=\frac{\pi}{4}.$$

Thus $\psi = \theta+\phi$

(i.e) $\psi = \frac{\pi}{2}+\frac{\pi}{4}$

(i.e) $\psi = \frac{3\pi}{4}$.

$\therefore$ Slope of the tangent $=\tan\psi$

$$= \tan\left(\frac{3\pi}{4}\right)$$

$$= \tan\left(\frac{\pi}{2}+\frac{\pi}{4}\right)$$

$$= -\tan\left(\frac{\pi}{4}\right) = -1$$

Check your progress :

Question :

Find the slope of the curve $r=a\sin 2\theta$ at $\frac{\pi}{4}$.

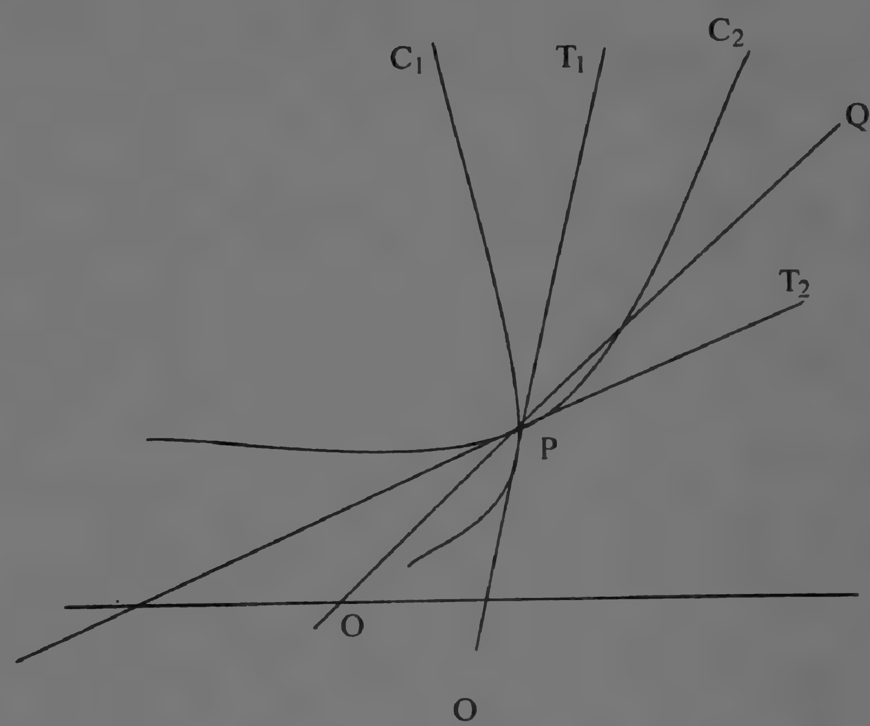
(**Answer** : slope of the tangent $= -1$)

Space for Hints

2.4. The angle of intersection of two curves

Book work : Find the angle of intersection of two curves in polar coordinates.

Solution :



Let C_1 and C_2 be two given curves.

Let $P(r,\theta)$ be a point of intersection of the curves.

From figure $T_2\hat{P}T_2 = Q\hat{P}T_1 - Q\hat{P}T_2$.

(i.e) $\psi = \phi - \phi'$

$\therefore \quad \tan\psi = \tan(\phi - \phi')$

$$= \frac{\tan\phi - \tan\phi'}{1 + \tan\phi \cdot \tan\phi'}$$

Because equation of the curves are known, then the coordinates of P can be found and then $\tan\phi$ and $\tan\phi'$ can be calculated from the equations of the curves.

Example 2.4.1 :

Find the angle on intersection of the curves $r = \dfrac{a}{1+\cos\theta}$ and $r = \dfrac{b}{1-\cos\theta}$.

Space for Hints

Solution :

Given that $r = \dfrac{a}{1+\cos\theta}$ -- (2.1)

and $r = \dfrac{b}{1-\cos\theta}$ -- (2.2)

Let ϕ and ϕ' be the angles, which the radius vector to the point intersection of curves, makes with the tangents to the curves at the point.

We want to find $\phi : \phi'$

For the curve (2.1), taking log on both sides, we get,

$\log r = \log a - \log(1+\cos\theta)$

Differentiate with respect to θ, we have,

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 - \frac{-\sin\theta}{1+\cos\theta}.$$

$$= \frac{\cancel{2}\sin(\theta/2)\cdot\cos(\theta/2)}{\cancel{2}\cos^2(\theta/2)}$$

$$= \tan(\theta/2)$$

$$\therefore \quad \frac{dr}{d\theta} = r\cdot\tan(\theta/2)$$

Thus $\tan\phi = r\dfrac{d\theta}{dr}$

$$= \frac{r}{r\cdot\tan(\theta/2)}$$

$$= \cot(\theta/2)$$

$$= \tan\left(\frac{3\pi}{2} - \frac{\theta}{2}\right)$$

$\therefore \quad \phi = \dfrac{3\pi}{2} - \dfrac{\theta}{2}$ --- (2.3)

Again for the curve (2.2) taking log on both sides, we get,

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$$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 - \frac{\sin\theta}{1-\cos\theta}$$

$$= -\frac{\cancel{2}\sin(\theta/2)\cdot\cos(\theta/2)}{\cancel{2}\sin^2(\theta/2)}$$

$$= -\cot(\theta/2)$$

$$\therefore \quad \frac{dr}{d\theta} = -r\cdot\cot(\theta/2)$$

Thus $\tan\phi' = r\dfrac{d\theta}{dr}$

$$= \frac{r}{r\cdot\left(-\cot(\theta/2)\right)}$$

$$= -\frac{1}{\cot(\theta/2)}$$

$$= -\tan(\theta/2)$$

$$= \tan\left(\pi - \frac{\theta}{2}\right)$$

$$\therefore \phi' = \pi - \frac{\theta}{2} \text{ -- (2.4)}$$

Now $(2.3)-(2.4) \Rightarrow \phi - \phi' = \left(\dfrac{3\pi}{2} - \dfrac{\theta}{2}\right) - \left(\pi - \dfrac{\theta}{2}\right)$

$$= \frac{3\pi}{2} - \cancel{\frac{\theta}{2}} - \pi + \cancel{\frac{\theta}{2}}$$

$$= \frac{\pi}{2}$$

Hence the curves cut each other at right angles

Example 2.3.2 :

Find the angle of intersection of the cardioids $r = a(1+\cos\theta)$ and $r = b(1-\cos\theta)$.

Solution : Given that $r = a(1+\cos\theta)$ ---------------------------------(2.5)

and $r = b(1-\cos\theta)$ ---------------------------------(2.6)

Let P be the point of intersection of (2.5) and (2.6).

Let PT and PT_1 be the tangents to the curves (2.5) and (2.6) at P.

Let ϕ and ϕ' be the angles, which the radius vector to the point intersection of curves, makes with the tangents to the curves at the point.

We want to find $\phi : \phi'$

Now $r = a(1+\cos\theta)$

Differentiate with respect to θ, we get,

$$\frac{dr}{d\theta} = -a\sin\theta$$

and $\tan\phi = r \cdot \dfrac{d\theta}{dr}$

$$= \frac{\cancel{a}(1+\cos\theta)}{(-\cancel{a}\sin\theta)}$$

$$= -\frac{\cancel{2}\cos^2(\theta/2)}{\cancel{2}\sin(\theta/2)\cos(\theta/2)}$$

$$= -\cot(\theta/2)$$

$$\therefore \ \phi = \frac{\pi}{2} + \frac{\theta}{2}$$

Again from (2), $r = b(1-\cos\theta)$

Differentiate with respect to θ, we get,

$$\frac{dr}{d\theta} = a\sin\theta$$

and $\tan\phi' = r \cdot \dfrac{d\theta}{dr}$

$$= \frac{\cancel{a}(1-\cos\theta)}{\cancel{a}\sin\theta}$$

$$= \frac{\cancel{2}\sin^2(\theta/2)}{\cancel{2}\sin(\theta/2)\cos(\theta/2)}$$

$$= \tan(\theta/2)$$

$$\therefore \quad \phi' = \frac{\theta}{2}$$

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Space for Hints

Thus $\phi - \phi' = \frac{\pi}{2} + \frac{\theta}{2} - \frac{\theta}{2} = \frac{\pi}{2}$

Hence the curves cut each other at right angles.

Example 2.3.3 :

Find the angle of intersection of the curves $r^n = a^n \cos n\theta$ and $r^n =$ $a^n \sin n\theta$

Solution : Given that $r^n = a^n \cos n\theta$ ---------------------------------- (2.7)

and $r^n = a^n \sin n\theta$ --- (2.8)

Let P be the point of intersection of (2.7) and (2.8).

Let PT and PT_1 be the tangents to the curves (2.7) and (2.8) at P.

Let ϕ and ϕ' be the angles between the radius vector at the point intersection of curves and the tangents to the curves respectively.

We want to find $\phi : \phi'$

Taking log for the curve (2.7) on both sides, we get,

$n \log r = n \log a + \log \cos n\theta$

Differentiate with respect to θ, we get,

$$\frac{n}{r} \cdot \frac{dr}{d\theta} = -\frac{n \sin n\theta}{\cos n\theta}$$

$$\frac{dr}{d\theta} = -\frac{r}{\cancel{n}} \cdot \cancel{n} \tan n\theta$$

(i.e) $$\frac{dr}{d\theta} = -r \cdot \tan n\theta$$

$$\therefore \tan\phi = r \cdot \frac{d\theta}{dr}$$

$$= -\frac{\cancel{r}}{\cancel{r} \tan n\theta}$$

$$= -\cot n\theta$$

$$= \tan\left(\frac{\pi}{2} + n\theta\right)$$

Thus $\phi = \frac{\pi}{2} + n\theta$

Again taking log on both sides for the curve (2.8), we get,

$n \log r = n \log a + \log \sin n\theta$

Differentiate with respect to θ, we get,

$$\frac{n}{r} \cdot \frac{dr}{d\theta} = \frac{n \cos n\theta}{\sin n\theta}$$

$$\frac{dr}{d\theta} = \frac{r}{\cancel{n}} \cdot \cancel{n} \cot n\theta$$

(i.e) $$\frac{dr}{d\theta} = r \cdot \cot n\theta$$

$$\therefore \tan \phi' = r \cdot \frac{d\theta}{dr}$$

$$= \frac{\cancel{r}}{\cancel{r} \cot n\theta}$$

$$= \tan n\theta$$

Thus $\phi' = n\theta$

Now $\phi - \phi' = \frac{\pi}{2} + n\theta - n\theta = \frac{\pi}{2}$

Hence the curves cut each other at right angles.

Check your progress

Question :

Find the angle of intersection of the curves $r = \sin\theta + \cos\theta$ and $r = 2\sin\theta$.

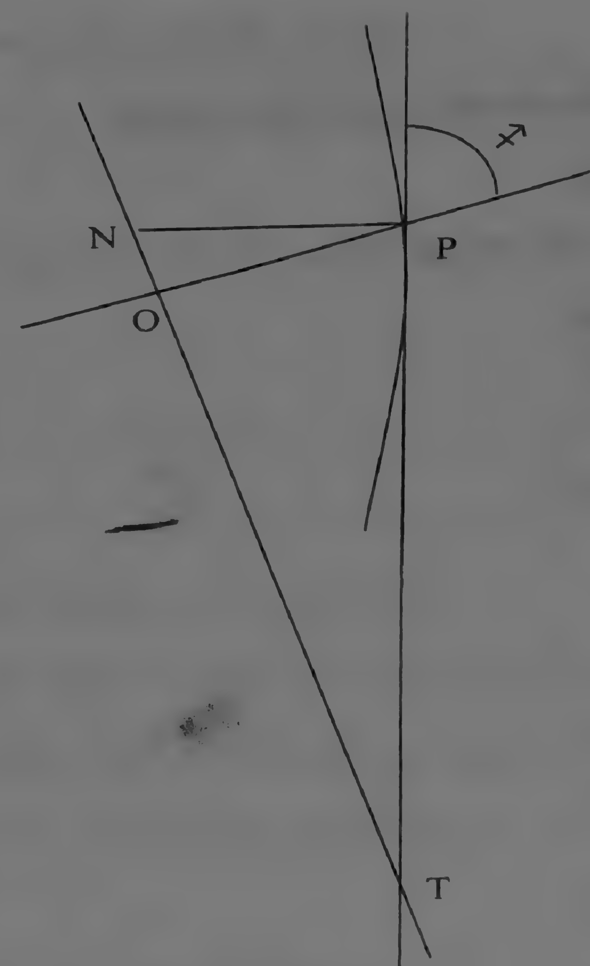
(Answer the angle of intersection is $\frac{\pi}{4}$)

2.5. Polar subtangent and polar subnormal

Bookwork : Find the equation of polar subtangent and polar subnormal.

Space for Hints

Space for Hints



Draw a line NT through the pole perpendicular to the radius vector of the point P on the curve. Let PT be the tangent and PN the normal to the curve at P. Then at the point P,

OT = length of the polar subtangent

ON = length of the polar subnormal.

and the length of the polar subnormal

= ON

$$= OP \cdot \tan O\hat{P}N$$

$$= OP \cdot \tan(T\hat{P}N - T\hat{P}O)$$

$$= OP \cdot \tan\left(\frac{\pi}{2} - \phi\right)$$

$$= OP \cdot \cot\phi$$

$$= \frac{r}{\tan\phi}$$

$$= \frac{\cancel{r}}{\cancel{r}\,\frac{d\theta}{dr}}$$

$$= \frac{dr}{d\theta}$$

Space for Hints

Example 2.5.1 :

Find the polar subtangent to the curve $\frac{2a}{r} = 1 - \cos\theta$

Solution : Given that $\frac{2a}{r} = 1 - \cos\theta$

Differentiate with respect to θ, we get, $2a\left(-\frac{1}{r^2}\right)\cdot\frac{dr}{d\theta} = \sin\theta$

$$\therefore \quad \frac{dr}{d\theta} = -\frac{r^2 \cdot \sin\theta}{2a}$$

Thus $\frac{d\theta}{dr} = -\frac{2a}{r^2 \cdot \sin\theta}$

Now $\tan\phi = r \cdot \frac{d\theta}{dr}$

$$= r \cdot \left(-\frac{2a}{r^2 \cdot \sin\theta}\right)$$

$$= -\frac{2a}{r \cdot \sin\theta}$$

$$= -\frac{(1 - \cos\theta)}{\sin\theta}$$

Hence the polar subtangent is $r^2 \frac{d\theta}{dr}$

$$= r \cdot r \cdot \frac{d\theta}{dr}$$

$$= r \cdot \tan\phi$$

$$= \frac{2a}{1 - \cos\theta} \cdot \left(-\frac{1 - \cos\theta}{\sin\theta}\right)$$

Space for Hints

$= -2a\operatorname{cosec}\theta$

$= 2a\operatorname{cosec}\theta$ (numerically)

Check your progress :

Question :

For the cardioid $a(1-\cos\theta)$, prove that the polar subtangent is

$$2a\sin^3\left(\frac{\theta}{2}\right)\cdot\tan\left(\frac{\theta}{2}\right).$$

Example 2.5.2 :

Find the polar subtangent and polar subnormal to the curve $r = a e^{\theta\cot\alpha}$.

Solution : Given that $r = a e^{\theta\cot\alpha}$.

$$\therefore \quad \frac{dr}{d\theta} = a e^{\theta\cot\alpha}\cot\alpha$$

(i.e) $\dfrac{dr}{d\theta} = r\cot\alpha$

Thus the polar subnormal $= r\cot\alpha$.

Again $\dfrac{d\theta}{dr} = \dfrac{1}{r\cot\alpha}$.

$$\therefore \quad r^2\frac{d\theta}{dr} = \frac{r^2}{r\cot\alpha}$$

$$= r\tan\alpha$$

Hence the polar subtangent $= r\tan\alpha$.

Example 2.5.2 :

For the curve $r = a\theta$, prove that the polar subtangent varies as the square of the radius vector and the polar subnormal is constant.

Solution : Given that $r = a\theta$.

Differentiate with respect to θ, we get, $\dfrac{dr}{d\theta} = a$, which is a constant.

Again $\frac{d\theta}{dr} = \frac{1}{a}$

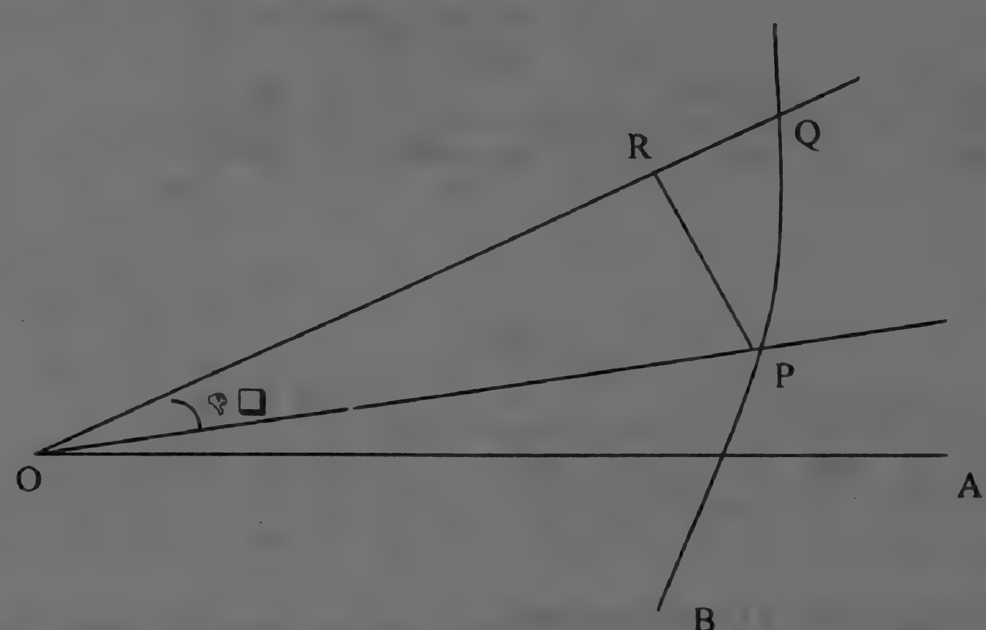
and hence $r^2 \frac{d\theta}{dr} = \frac{r^2}{a}$

Thus the polar subtangent varies as

Thus the polar subtangent varies as r^2 .

This proves the problem.

2.6. The length of arc in polar coordinates



Let O be the pole and OA be the initial line.

Let P, Q be two neighboring points on the curve and let $P(r,\theta)$ and $Q(r+\Delta r, \theta + \Delta\theta)$ be a coordinates of P, Q respectively.

Clearly $P\hat{O}R = \Delta\theta$.

Let B be fixed point on the curve.

Let $\overset{\frown}{BP} = s$ and $\overset{\frown}{BQ} = s + \Delta s$.

Thus $\overset{\frown}{PQ} = \Delta s$.

Now from ΔOPR , $PR = OP \cdot \sin\Delta\theta = r\sin\Delta\theta$

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and $OR = OP \cdot \cos\Delta\theta = r\cos\Delta\theta$.

Again $QR = OQ - OR$

$$= r + \Delta r - r\cos\Delta\theta$$

$$= r(1-\cos\Delta\theta) + \Delta r$$

$$= 2r\sin^2\left(\frac{\Delta\theta}{2}\right) + \Delta r$$

Since Δs is small then we consider $\overset{\frown}{PQ} = PQ$.

$\therefore$ from ΔPQR, $PQ^2 = PR^2 + RQ^2$

$$= (r\sin\Delta\theta)^2 + \left(2r\sin^2\left(\frac{\Delta\theta}{2}\right) + \Delta r\right)^2$$

Thus $$\left(\frac{PQ}{\Delta\theta}\right)^2 = \left(r\frac{\sin\Delta\theta}{\Delta\theta}\right)^2 + \left(2r\frac{\sin^2\left(\frac{\Delta\theta}{2}\right)}{\Delta\theta} + \frac{\Delta r}{\Delta\theta}\right)^2$$

$$= r^2\left(\frac{\sin\Delta\theta}{\Delta\theta}\right)^2 + \left(r\cdot\frac{\sin\left(\frac{\Delta\theta}{2}\right)}{\frac{\Delta\theta}{2}}\cdot\sin\left(\frac{\Delta\theta}{2}\right) + \frac{\Delta r}{\Delta\theta}\right)^2$$

If $Q \to P$, then $\Delta\theta \to 0$ and we have

$$\frac{\sin\Delta\theta}{\Delta\theta} \to 1, \quad \frac{\Delta r}{\Delta\theta} \to \frac{dr}{d\theta} \quad \text{and} \quad \sin\left(\frac{\Delta\theta}{2}\right) \to 0.$$

$$\therefore \quad \frac{PQ}{\Delta\theta} = \frac{PQ}{\overset{\frown}{PQ}}\cdot\frac{\overset{\frown}{PQ}}{\Delta\theta} \to 1\cdot\frac{ds}{d\theta} = \frac{ds}{d\theta}$$

Thus $$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

Hence $$\boxed{\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}}$$

Note 1 : We shall prove

$$\frac{ds}{dr} = \sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}$$

Proof of note 1 : In proof of above bookwork, we have,

$$(PQ)^2 = (r\sin\Delta\theta)^2 + \left(2r\sin^2\left(\frac{\Delta\theta}{2}\right)+\Delta r\right)^2$$

(i.e) $(\Delta s)^2 = (r\sin\Delta\theta)^2 + \left(2r\sin^2\left(\frac{\Delta\theta}{2}\right)+\Delta r\right)^2$

Divide by $(\Delta r)^2$ on both sides, we get,

$$\left(\frac{\Delta s}{\Delta r}\right)^2 = \left(\frac{r\sin\Delta\theta}{\Delta r}\right)^2 + \left(1+2\frac{r}{\Delta r}\sin^2\left(\frac{\Delta\theta}{2}\right)\right)^2$$

Now $\left(\frac{PQ}{\Delta s}\right)^2\cdot\left(\frac{\Delta s}{\Delta r}\right)^2$

$$= r^2\cdot\left(\frac{\sin\Delta\theta}{\Delta\theta}\right)^2\cdot\left(\frac{\Delta\theta}{\Delta r}\right)^2 + \left[1+r\cdot\frac{\sin(\Delta\theta/2)}{\Delta\theta/2}\cdot\sin\left(\frac{\Delta\theta}{2}\right)\cdot\frac{\Delta\theta}{\Delta r}\right]^2$$

If $Q\to P$, then $\Delta\theta\to 0$ and we have

$\frac{\sin\Delta\theta}{\Delta\theta}\to 1$, $\frac{\Delta\theta}{\Delta r}\to\frac{d\theta}{dr}$ and $\sin\left(\frac{\Delta\theta}{2}\right)\to 0$.

Now (1) $\Rightarrow$ $1\cdot\left(\frac{ds}{dr}\right)^2 = r^2\cdot 1\cdot\left(\frac{d\theta}{dr}\right)^2 + \left[1+r\cdot 1\cdot 0\cdot\frac{d\theta}{dr}\right]^2$

$$\left(\frac{ds}{dr}\right)^2 = 1+r^2\cdot\left(\frac{d\theta}{dr}\right)^2.$$

Thus $\boxed{\frac{ds}{dr} = \sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}}$

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Note 2 : we shall note that $\sin\phi = r\dfrac{d\theta}{ds}$ and $\cos\phi = \dfrac{dr}{ds}$.

Proof of note 2 : We know that $\tan\phi = r\dfrac{d\theta}{dr}$

Thus $$\sin\phi = \frac{1}{\operatorname{cosec}\phi}$$

$$= \frac{1}{\sqrt{1+\cot^2\phi}}$$

$$= \frac{\tan\phi}{\sqrt{\tan^2\phi+1}}$$

$$= \frac{r\dfrac{d\theta}{dr}}{\sqrt{r^2\left(\dfrac{d\theta}{dr}\right)^2+1}}$$

$$= \frac{1}{\dfrac{d\theta}{dr}}\frac{r\dfrac{d\theta}{dr}}{\sqrt{r^2+\left(\dfrac{dr}{d\theta}\right)^2}}$$

$$= \frac{r}{\dfrac{ds}{d\theta}}$$

$$= r\frac{d\theta}{ds}.$$

and $$\cos\phi = \frac{1}{\sec\phi}$$

$$= \frac{1}{\sqrt{1+\tan^2\phi}}$$

$$= \frac{1}{\sqrt{1+r^2\left(\dfrac{d\theta}{dr}\right)^2}}$$

$$= \frac{1}{ds/dr} = \frac{dr}{ds}.$$

Example 2.6.1 :

Find $\frac{ds}{d\theta}$ for the curve $r^2 = a^2 \cos 2\theta$.

Solution : Given that $r^2 = a^2 \cos 2\theta$.

Taking log on both sides, we get, $2\log r = 2\log a + \log\cos 2\theta$

Differentiate with respect to θ, we get, $\frac{2}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{1}{\cos 2\theta}\cdot(-2\sin 2\theta)$

(i.e) $\frac{dr}{d\theta} = -\frac{r}{2}\cdot\frac{2\sin 2\theta}{\cos 2\theta}$

(i.e) $\frac{dr}{d\theta} = -r\cdot\tan 2\theta$

Thus $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$

$= \sqrt{r^2 + \tan^2 2\theta}$

$= r\sec 2\theta$

$= \frac{r}{\cos 2\theta}$

$= r\cdot\frac{a^2}{r^2}$

$= \frac{a^2}{r}$

Example 2.6.2 :

For the curve $r^n = a^n \cos n\theta$, prove that $\frac{ds}{d\theta}$ varies inversely as the $(n-1)^{th}$ power of r.

Proof : Given that $r^n = a^n \cos n\theta$

Taking log on both sides, we have,

$n\log r = n\log a + \log\cos n\theta$

Differentiate with respect to θ, we get,

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$$\frac{n}{r}\cdot\frac{dr}{d\theta}=0+\frac{1}{\cos n\theta}\cdot(-n\sin n\theta)$$

(i.e) $$\frac{dr}{d\theta}=-\frac{r}{\cancel{n}}\cdot\frac{\cancel{n}\sin n\theta}{\cos n\theta}$$

(i.e) $$\frac{dr}{d\theta}=-r\cdot\tan n\theta$$

Thus $$\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$$

$$= \sqrt{r^2+\tan^2 n\theta}$$

$$= r\sec n\theta$$

$$= \frac{r}{\cos n\theta}= r\cdot\frac{a^n}{r^n} = \frac{a^n}{r^{n-1}} \propto \frac{1}{r^{n-1}}$$

(i.e) $\frac{ds}{d\theta}$ varies inversely as the $(n-1)^{th}$ power of r.

This proves the problem.

Summary

In this unit we have leaned how to find the subtangent and subnormal, the concepts of polar coordinates, how to find the angle between two curves, how to find polar subtangent and subnormal and the method of finding length of arc.

Further Reading

You can also refer the following books for further reading.

(1) Calculus by Arumugam and Isaac

(2) Differential Calculus by Shanti Narayanan

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UNIT III

CURVATURES AND EVOLUTES

Introduction

Unit Objectives

Unit Structure

Check your progress

Summary

Further Reading

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Objectives :

In this unit we are going to discuss Envelopes, Curvatures, Radius of curvature, Centre of curvature and Evolutes.

After completing this unit, students may able to know

- Envelopes
- Curvatures
- Radius of curvature
- Centre of curvature
- Evolutes.

Introduction

One of applications of differential calculus is curvatures and evolutes. A separate branch of differential geometry is available to study curvature and envelopes.

3.1. Envelope

3.1.1 Family of curves

Consider the equation $(x-a)^2 + y^2 = r^2$.

The above equation represents a circle of radius r and centre $(a,0)$.

By fixing the radius r and allow to vary a, we get, lot of circles, each circle having radius r and the centre lies on x – axis.

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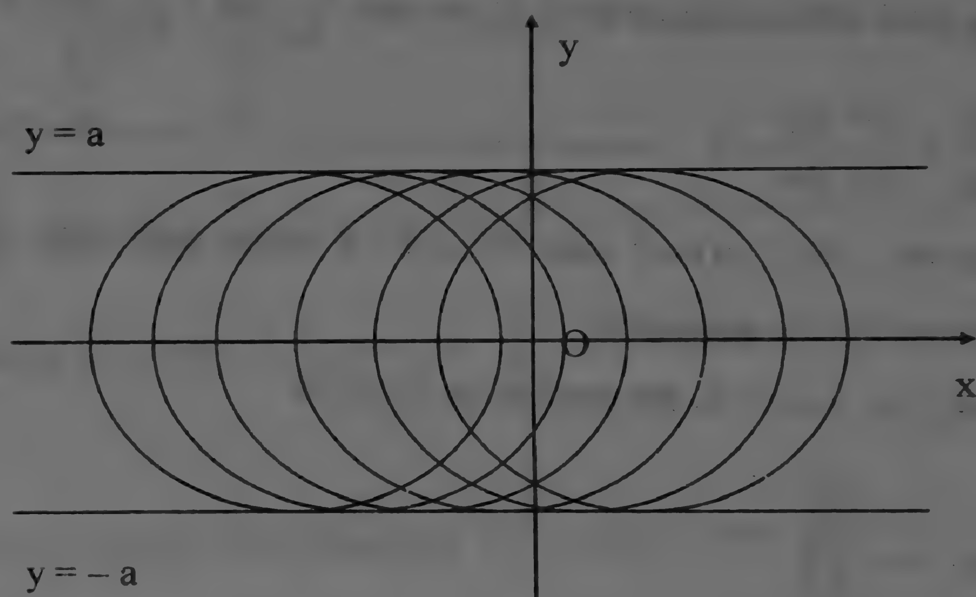

The totality or series of curves formed in this way is called a *family of curves* and the quantity a whose different values gives different members of the family is called *parameter* of the family of curves.

Definition : (Envelope)

The envelope of a family of curves is the locus of the limiting positions of the point of intersection of any two consecutives members of the family, when one of them tends to coincide with the other which kept fixed.

3.1.2 Methods of finding the envelope

Let the family of curves C be $f(x,y,t)=0$ and let us assume that a curve E, the envelope of the family exists and its equation be $F(x,y)=0$.

Let us further assume that for a particular value of t, say α it touches E at (ξ,η).

Therefore $f(\xi,\eta,\alpha)=0$ -- (3.1)

and $F(\xi,\eta)=0$ -- (3.2)

assuming ξ,η,α are independent variables and taking total differentiation in (1), we have, $\dfrac{\partial f}{\partial \xi}d\xi+\dfrac{\partial f}{\partial \eta}d\eta+\dfrac{\partial f}{\partial \alpha}d\alpha=0$

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(i.e) $\dfrac{\partial f}{\partial \xi}\dfrac{d\xi}{d\alpha}+\dfrac{\partial f}{\partial \eta}\dfrac{d\eta}{d\alpha}+\dfrac{\partial f}{\partial \alpha}=0$ -- (3.3)

Again taking total differentiation in (3.2), we get, $\dfrac{\partial F}{\partial \xi}d\xi+\dfrac{\partial F}{\partial \eta}d\eta=0$

(i.e) $\dfrac{\partial F}{\partial \xi}\dfrac{d\xi}{d\alpha}+\dfrac{\partial F}{\partial \eta}\dfrac{d\eta}{d\alpha}=0$ --(3.4)

Since the curves $f(x,y,\alpha)=0$ and $F(x,y)=0$ touch each other at (ξ,η), their gradients at (ξ,η) are equal.

For the curve $f(x,y,\alpha)=0$, the gradient at (ξ,η) is

$$\frac{dx}{dy} \text{ at } (\xi,\eta) = -\frac{\dfrac{\partial f}{\partial \xi}}{\dfrac{\partial f}{\partial \eta}}$$

Similarly, for the curve $F(x,y)=0$, the gradient at (ξ,η) is

$$\frac{dx}{dy} \text{ at } (\xi,\eta) = -\frac{\dfrac{\partial F}{\partial \xi}}{\dfrac{\partial F}{\partial \eta}}$$

Hence $-\dfrac{\dfrac{\partial f}{\partial \xi}}{\dfrac{\partial f}{\partial \eta}} = -\dfrac{\dfrac{\partial F}{\partial \xi}}{\dfrac{\partial F}{\partial \eta}}$

(i.e) $\dfrac{\dfrac{\partial f}{\partial \xi}}{\dfrac{\partial f}{\partial \eta}} = \dfrac{\dfrac{\partial F}{\partial \xi}}{\dfrac{\partial F}{\partial \eta}}$ -- (3.5)

From (3.4), $\dfrac{\dfrac{\partial F}{\partial \xi}}{\dfrac{\partial F}{\partial \eta}} = -\dfrac{\dfrac{\partial \eta}{\partial \alpha}}{\dfrac{\partial \xi}{\partial \alpha}}$ -- (3.6)

Thus from (3.5) and (3.6), we have,

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$$\frac{\frac{\partial f}{\partial \xi}}{\frac{\partial f}{\partial \eta}} = -\frac{\frac{\partial \eta}{\partial \alpha}}{\frac{\partial \xi}{\partial \alpha}}$$

(i.e) $\frac{\partial f}{\partial \xi}\cdot\frac{\partial \xi}{\partial \alpha}+\frac{\partial f}{\partial \eta}\cdot\frac{\partial \eta}{\partial \alpha}=0$ -- (3.7)

Comparing (3.3) and (3.7), we get, $\frac{\partial f}{\partial \alpha}=0$ and this equation is satisfied by (ξ,η).

Now (ξ,η) satisfies the equations $f(x,y,\alpha)=0$ and $\frac{\partial f}{\partial \alpha}=0$.

Hence the envelope of the family of curves $f(x,y,t)=0$ is got by eliminating t between the equations $f(x,y,t)=0$ and $\frac{\partial}{\partial \xi}(f(x,y,t))=0$.

Example 3.1.1 :

Find the envelope of $f(x,y,t)=0$ which is quadratic in t.

Solution : Given that the family of curves are $f(x,y,t)=0$ which is quadratic in t.

Let $f(x,y,t) = at^2+bt+c=0$ --- (3.8)

where a,b,c are functions of x and y.

Differentiate (3.8) partially with respect to t, we get, $2at+b=0$

(i.e) $2at=-b$

(i.e) $t=-\frac{b}{2a}$ --- (3.9)

Eliminating t between (3.8) and (3.9), we get, $a\left(-\frac{b}{2a}\right)^2+b\left(-\frac{b}{2a}\right)+c=0$

(i.e) $a\left(\frac{b^2}{4a^2}\right)-\left(\frac{b^2}{2a}\right)+c=0$

(i.e) $\frac{b^2}{4a}-\frac{b^2}{2a}+c=0$

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(i.e) $\dfrac{b^2-4ac}{4a}=0$

(i.e) $b^2-4ac=0$

which gives the equation of envelope of (3.8).

Example 3.1.2 :

Find the envelope of the following family of straight lines :

(1) $y=mx+\dfrac{a}{m}$,

(2) $y=mx\pm\sqrt{a^2m^2+b^2}$

Solution of (1) : Given that $y=mx+\dfrac{a}{m}$ ------------------------------ (3.10)

(i.e) $ym=m^2x+a$

(i.e) $xm^2-ym+a=0$ -- (3.11)

which quadratic in m

Thus the envelope of (3.10) are obtained from (3.11) by equating the discriminent to zero.

(i.e) $(-y)^2-4xa=0$

(i.e) $y^2=4xa$ which is the required envelope of (3.10).

Solution of (2) : Given that $y=mx\pm\sqrt{a^2m^2+b^2}$ ----------------- (3.12)

(i.e) $y-mx=\pm\sqrt{a^2m^2+b^2}$

Squaring on both sides, we get,

$$(y-mx)^2=\left(\pm\sqrt{a^2m^2+b^2}\right)^2$$

$$y^2+m^2x^2-2ymx=a^2m^2+b^2$$

(i.e) $(a^2-x^2)m^2-2xym+(b^2-y^2)=0$ --------------------------- (3.13)

which quadratic in m

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Thus the envelope of (3.12) are obtained from (3.13) by equating the discriminent to zero.

(i.e) $(-2xy)^2 - 4(a^2 - x^2)(b^2 - y^2) = 0$

(i.e) $\cancel{4}x^2y^2 - \cancel{4}(a^2b^2 - a^2y^2 - b^2x^2 + x^2y^2) = 0$

(i.e) $x^2y^2 - a^2b^2 + a^2y^2 + b^2x^2 - x^2y^2 = 0$

(i.e) $a^2y^2 + b^2x^2 = a^2b^2$

(i.e) $\dfrac{a^2}{x^2} + \dfrac{b^2}{y^2} = 1$

which is the required envelope of (3.12).

Example 3.1.3 :

Find the envelope of the family of a straight lines $y + tx = 2at + at^3$.

Solution : Given that $y + tx = 2at + at^3$ -------------------------------- (3.14)

Differentiate with respect to *t*, we have, $x = 2a + 3at^2$ --------------- (3.15)

To get envelopes of (3.14), we have to eliminate *t* from (3.14) and (3.15).

From (3.15) (2) we have, $3at^2 = x - 2a$

(i.e) $t^2 = \dfrac{x-2a}{3a}$ -- (3.16)

From (3.14), $y = 2at + at^3 - tx$

(i.e) $y = t(2a + at^2 - x)$ -- (3.17)

From (3.16) and (3.17), we have,

$$y = t\left(2a + \cancel{a}\,\frac{x-2a}{3\cancel{a}} - x\right)$$

(i.e) $y = t\left(\dfrac{6a + x - 2a - 3x}{3}\right)$

(i.e) $y = t\left(\dfrac{4a - 2x}{3}\right)$

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(i.e) $y = \frac{2t}{3}(2a - x)$

Squaring on both sides, we get, $y^2 = \frac{4t^2}{9}(2a - x)^2$ ------------------ (3.18)

From (3.16) and (3.18), we have,

$$y^2 = \frac{4}{9}\left(\frac{x-2a}{3a}\right)(2a - x)^2$$

(i.e) $y^2 = \frac{4}{9}\left(\frac{x-2a}{3a}\right)(x - 2a)^2$

(i.e) $27y^2 = 4(x - 2a)^3$

which is the equation of the required envelope of (3.14).

Example 3.1.4 :

Find the envelope of the family of circles $(x-a)^2 + y^2 = 2a$.

Solution : Given that $(x-a)^2 + y^2 = 2a$ ------------------------------ (3.19)

Differentiate partially with respect to a, we have,

$$2(x-a)(-1) + 0 = 2$$

(i.e) $-(x-a) = 1$

(i.e) $a = x + 1$ -- (3.20)

Eliminating a between (3.19) and (3.20), we get,

$$(-1)^2 + y^2 = 2(x+1)$$

(i.e) $1 + y^2 = 2x + 2$

(i.e) $y^2 = 2x + 1$

which is the required equation of envelope of (3.19).

Example 3.1.5 :

Find the envelope of the family of curves $\frac{x^2}{a^2} + \frac{y^2}{k^2 - a^2} = 1$, where a is a parameter.

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Solution : Given that $\frac{x^2}{a^2}+\frac{y^2}{k^2-a^2}=1$ ------------------------------ (3.21)

(i.e) $x^2(k^2-a^2)+y^2a^2=a^2(k^2-a^2)$

(i.e) $a^4-a^2(x^2-y^2+k^2)+x^2k^2=0$ ------------------------------ (3.22)

Since (3.22) is quadratic in a^2, its eliminant is $B^2-2AC=0$.

(i.e) $(x^2-y^2+k^2)^2-4(1)(x^2k^2)=0$

(i.e) $(x^2-y^2+k^2)^2=4x^2k^2$

(i.e) $x^2-y^2+k^2=\pm\, 2xk$

(i.e) $x^2\ \mathrm{m}2xk+k^2=y^2$

(i.e) $(x\,\mathrm{m}2k)^2=y^2$

(i.e) $x\pm 2k=\pm y$

(i.e) $x\pm y=\pm 2k$

which are the required equations of envelope of (3.21).

Example 3.1.6 :

Find the envelope of the family of the straight lines $\frac{x}{a}+\frac{y}{b}=1$ where the parameters are related by the equation $a^2+b^2=c^2$ where c is a constant.

Solution : Given that $\frac{x}{a}+\frac{y}{b}=1$ -- (3.23)

and the parameters are connected by $a^2+b^2=c^2$ ------------------- (3.24)

Assume that a and b are function of t.

Differentiate (3.23) with respect to t, we get,

$$-\frac{x}{a^2}\frac{da}{dt}-\frac{y}{b^2}\frac{db}{dt}=0 \quad \text{-- (3.25)}$$

Again differentiate (3.24) with respect to t, we get,

$$2a\frac{da}{dt}+2b\frac{db}{dt}=0 \quad \text{--- (3.26)}$$

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Comparing (3.25) and (3.26), we have, $\dfrac{x}{a^3}=\dfrac{y}{b^3}$ -------------------- (3.27)

Now we shall eliminate a and b from (3.23), (3.24) and (3.27), we have,

from (3.27), $\dfrac{\frac{x}{a}}{a^2}=\dfrac{\frac{y}{b}}{b^2}=\dfrac{\frac{x}{a}+\frac{y}{b}}{a^2+b^2}=\dfrac{1}{c^2}$ ------------------------------- (3.28)

(i.e) $a^3=c^2x$ and $b^3=c^2y$

Substituting the values of a and b in (3.24), we have,

$$\left(c^2x\right)^{2/3}+\left(c^2y\right)^{2/3}=c^2$$

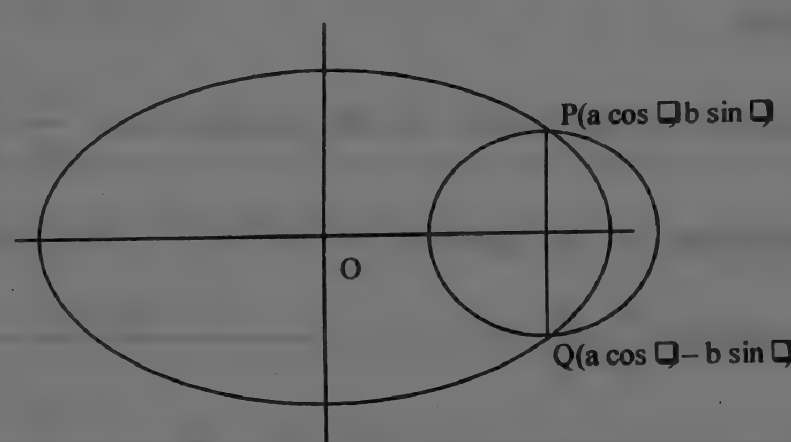
(i.e) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$

which is the required equation of envelope.

Example 3.1.7 :

Find the envelope of the circles whose diameters are double ordinates of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Solution :



Let the one end of the double ordinate be $(a\cos\theta,\ b\sin\theta)$.

Then the other end of the double ordinate is $(a\cos\theta,\ -b\sin\theta)$.

The equation of the circle whose diameter is PQ is

$$(x-a\cos\theta)(x-a\cos\theta)+(y-b\sin\theta)(y+b\sin\theta)=0$$

(i.e) $x^2+y^2-2ax\cos\theta+a^2\cos^2\theta-b^2\sin^2\theta=0$

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(i.e) $(a^2+b^2)\cos^2\theta - 2ax\cos\theta + (x^2+y^2-b^2) = 0$

which is quadratic in $\cos\theta$.

Thus equating the discriminent to zero, we get,

$4a^2x^2 - 4(a^2+b^2)(x^2+y^2-b^2) = 0$

(i.e) $\cancel{4}a^2x^2 - \cancel{4}(a^2x^2 + a^2y^2 - a^2b^2 + b^2x^2 + b^2y^2 - b^4) = 0$

(i.e) $a^2x^2 - a^2x^2 - a^2y^2 + a^2b^2 - b^2x^2 - b^2y^2 + b^4 = 0$

(i.e) $b^2x^2 + (a^2+b^2)y^2 = a^2b^2 + b^4$

(i.e) $b^2x^2 + (a^2+b^2)y^2 = b^2(a^2+b^2)$

(i.e) $\dfrac{x^2}{(a^2+b^2)} + \dfrac{y^2}{b^2} = 1$

which is the required envelope.

Check your progress

Questions :

(1) Find the envelope of the family of $x\cos^3\theta + y\sin^3\theta = a$

(2) Show that the envelope of the family of circles whose diameters are the double ordinates of the parabola $y^2 = 4ax$ is the parabola $y^2 = 4a(x+a)$.

(3) Find the envelope of family of ellipses $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ such that $a^2 + b^2 = c^2$.

3.2. Curvature

A curve has a definite direction at every point on it. At any particular point, the direction of the curve is the same as that of any tangent to the curve at that point. The directions usually changes from point to point and the tangent line rotates as the point moves along the curve.

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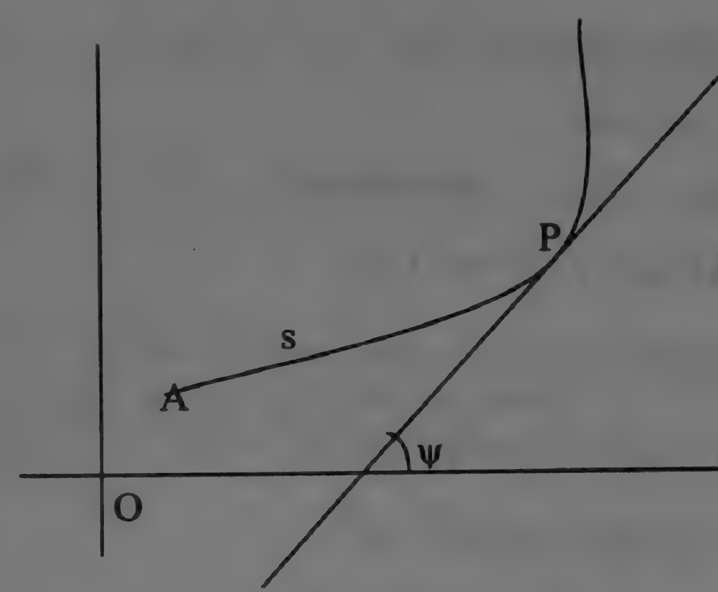
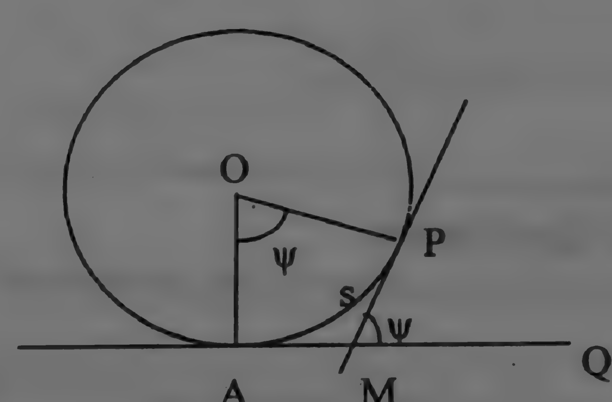


Let s denote the length of $\overset{\frown}{PQ}$ measured from some fixed point A on the curve and ψ the angle which the tangent makes with the x - axis. As P moves along the curve, s and ψ vary and the rate at which ψ increases relative to s (i.e) $\frac{d\psi}{ds}$ is called the curvature of the curve at the point P.

Example 3.2.1 :

Prove that curvature of the circle is the reciprocal of its radius.

Proof : Let r be the radius of a circle.



Draw AQ the tangent at A and O be the centre of the circle.

Space for Hints

Let P be any point on the circle and draw PM the tangent at P and which cuts AQ at M.

Let PM makes an angel ψ with AQ.

From figure $A\hat{O}P = \psi$

(i.e) $s = r\psi$

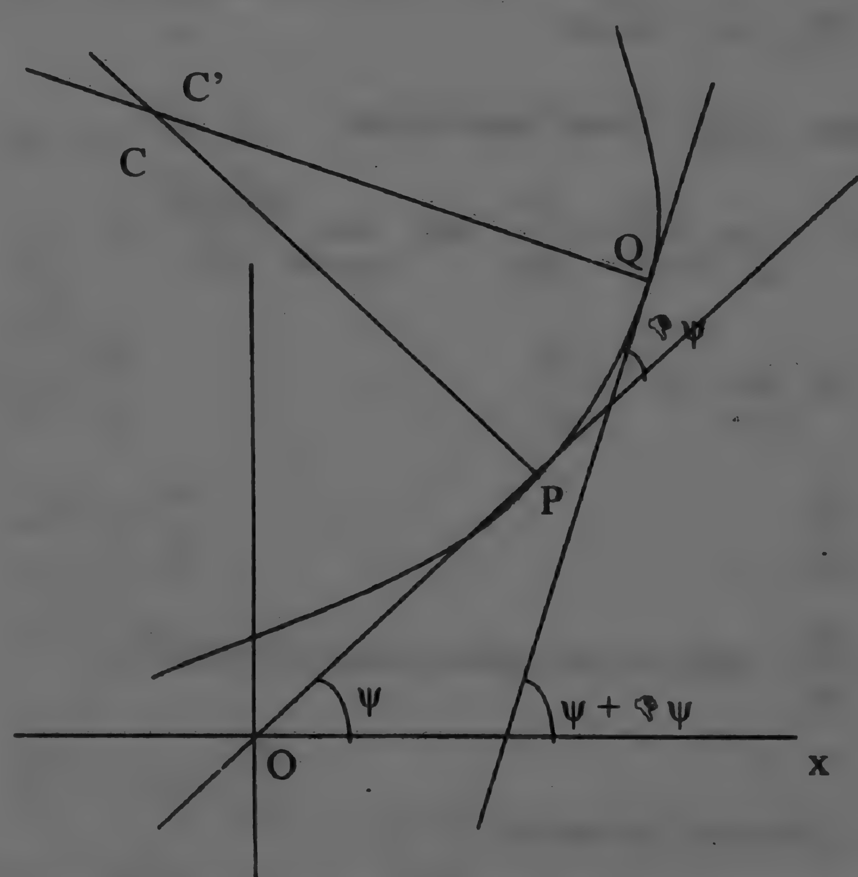
$\therefore \quad \dfrac{ds}{d\psi} = r$

(i.e) $\dfrac{d\psi}{ds} = \dfrac{1}{r}$

(i.e) the curvature of the circle is the reciprocal of the radius.

This proves the problem.

3.3. Circle, radius of curvature



Space for Hints

Let P be a point on a plane curve and Q be its neighboring point on the curve.

Let ψ and $\psi + \Delta\psi$ be the angles made by the tangents at P and Q with x - axis.

Let A be fixed point on the curve.

Let $\overset{\frown}{AP} = s$ and $\overset{\frown}{PQ} = \Delta s$.

Let the normals at P and Q intersect at C.

From the figure, it is clear that $\angle PCQ = \Delta\psi$.

$$\text{Now } \frac{PC'}{\sin P\hat{Q}C'} = \frac{PQ}{\sin P\hat{C}'Q}$$

$$= \frac{PQ}{\overset{\frown}{PQ}} \cdot \frac{\overset{\frown}{PQ}}{\sin P\hat{C}'Q}$$

$$= \frac{PQ}{\overset{\frown}{PQ}} \cdot \frac{\Delta s}{\sin \Delta\psi}$$

$$= \frac{PQ}{\overset{\frown}{PQ}} \cdot \frac{\Delta s}{\Delta\psi} \cdot \frac{\Delta\psi}{\sin\Delta\psi} \quad \text{--} \ (3.29)$$

If $Q \to P$ then $\angle PQC' \to 90^{\circ}$ and further

$$\lim_{Q\to P} \frac{PQ}{\overset{\frown}{PQ}} = 1, \quad \lim_{\Delta\psi\to 0} \frac{\Delta s}{\Delta\psi} = \frac{ds}{d\psi} \quad \text{and} \quad \lim_{\Delta\psi\to 0} \frac{\sin\Delta\psi}{\Delta\psi} = 1$$

Thus (3.29) $\Rightarrow \lim_{Q\to P} PC' = \dfrac{ds}{d\psi}$.

Thus $PC = \dfrac{ds}{d\psi}$

(i.e) $\dfrac{1}{PC} = \dfrac{d\psi}{ds}$ which is the radius of curvature at P.

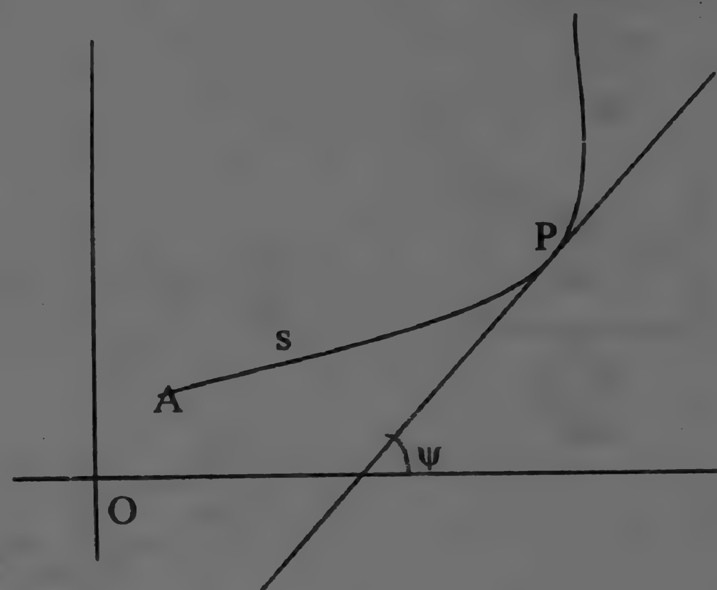
Definition : (Circle of curvature)

A circle is called the Circle of curvature at P if a circle which touches the given curve at the point has a radius equal to the radius of curvature at the point and lies on the same side of the tangent as the curve.

Space for Hints

3.3.1 Cartesian formula for the radius of curvature

Now we shall find the cartesian formula for the radius of curvature at a point on the given curve.



We know that $\frac{dy}{dx} = \tan\psi$.

Thus $\frac{d^2y}{dx^2} = \sec^2\psi \cdot \frac{d\psi}{dx}$

$$= \sec^2\psi \cdot \frac{d\psi}{ds} \cdot \frac{ds}{dx}$$

$$\therefore \quad \frac{d\psi}{ds} = \frac{\frac{d^2y}{dx^2}}{\sec^2\psi \cdot \frac{ds}{dx}}$$

Thus $\frac{ds}{d\psi} = \frac{\sec^2\psi \cdot \frac{ds}{dx}}{\frac{d^2y}{dx^2}}$

$$= \frac{\sec^2\psi \cdot \sec\psi \frac{ds}{dx}}{\frac{d^2y}{dx^2}} \text{ (since } \frac{dx}{ds} = \cos\psi \text{)}$$

Space for Hints

$$= \frac{\sec^3 \psi}{\frac{d^2 y}{dx^2}}$$

$$= \frac{(1+\tan\psi)^{\frac{3}{2}}}{\frac{d^2 y}{dx^2}}$$

$$= \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2 y}{dx^2}}$$

Thus the radius of curvature ρ can obtained from $\dfrac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2 y}{dx^2}}$

(i.e) $\rho = \dfrac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2 y}{dx^2}}$.

(i.e) $\rho = \dfrac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2}$.

Example 3.3.1 :

Find the radius of curvature of the curve $x^4 + y^4 = 2$ at $(1,1)$.

Solution : Given that $x^4 + y^4 = 2$ -------------------------------------- (3.30)

Differentiate (3.30) twice with respect to x, we get,

$$4x^3 + 4y^3 \frac{dy}{dx} = 0$$

Space for Hints

(i.e) $x^3 + y^3 \frac{dy}{dx} = 0$

(i.e) $y^3 \frac{dy}{dx} = -x^3$

(i.e) $\frac{dy}{dx} = -\frac{x^3}{y^3}$

and $$\frac{d^2y}{dx^2} = -\frac{y^3 3x^2 - x^3 3y^2 \frac{dy}{dx}}{y^6}$$

$$= -\frac{3x^2y^2\left(1 - x\frac{dy}{dx}\right)}{y^6}$$

$$= \frac{3x^2\left(x\frac{dy}{dx} - 1\right)}{y^4}$$

At $(1,1)$, $\frac{dy}{dx} = -1$ and $\frac{d^2y}{dx^2} = -6$

$$\therefore \quad \rho = \frac{\left(1 + y_1^2\right)^{\frac{3}{2}}}{y_2}$$

(i.e) $\rho = \frac{(1+1)^{\frac{3}{2}}}{-6}$

(i.e) $\rho = -\frac{\sqrt{2}}{3}$.

Example 3.3.2 :

Show that the radius of curvature at any point of the catenary $y = c\cosh\left(\frac{x}{c}\right)$ is equal to the length of the portion of the normal intercepted between the curve and the axis of x.

Space for Hints

Solution : Given that $y = c\cosh\left(\frac{x}{c}\right)$ ---------------------------------- (3.31)

$$\therefore \quad \frac{dy}{dx} = \sinh\left(\frac{x}{c}\right)$$

Thus $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = \left[1+\sinh^2\left(\frac{x}{c}\right)\right]^{\frac{3}{2}}$

$$= \left[\cosh^2\left(\frac{x}{c}\right)\right]^{\frac{3}{2}}$$

$$= \cosh^3\left(\frac{x}{c}\right)$$

Again $\frac{d^2y}{dx^2} = \frac{1}{c}\cosh\left(\frac{x}{c}\right)$.

Hence $\rho = \frac{\left(1+y_1^2\right)^{\frac{3}{2}}}{y_2}$

$$= \frac{\cosh^3\left(\frac{x}{c}\right)}{\frac{1}{c}\cosh\left(\frac{x}{c}\right)}$$

$$= c\cosh^2\left(\frac{x}{c}\right)$$

$$= \frac{c^2\cosh^2\left(\frac{x}{c}\right)}{c}$$

$$= \frac{y^2}{c} \quad \text{-- (3.32)}$$

Now the length of the normal $= y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}$

Space for Hints

$$= y\left[1+\sinh^2\left(\frac{x}{c}\right)\right]^{\frac{1}{2}}$$

$$= y\left[\cosh^2\left(\frac{x}{c}\right)\right]^{\frac{1}{2}}$$

$$= y\cosh\left(\frac{x}{c}\right)$$

$$= \frac{y^2}{c} \quad \text{------------------------------------} \quad (3.33)$$

From (3.32) and (3.33), we have, radius of curvature = length of the normal.

Example 3.3.3 :

If a curve is defined by the parametric equation $x = f(\theta)$ and $y = g(\theta)$ prove that the curvature is given by $\frac{1}{\rho} = \frac{x'y'' - y'x''}{\left(x'^2 + y'^2\right)^{3/2}}$ where $x' = \frac{dx}{d\theta}$, $y' = \frac{dy}{d\theta}$, $x'' = \frac{d^2x}{d\theta^2}$ and $y'' = \frac{d^2y}{d\theta^2}$.

Proof : We know that $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

(i.e) $\frac{dy}{dx} = \frac{y'}{x'}$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y'}{x'}\right)$$

$$= \frac{d}{d\theta}\left(\frac{y'}{x'}\right)\cdot\frac{d\theta}{dx}$$

$$= \frac{y''x' - y'x''}{x'^2}\cdot\frac{1}{x'}$$

Space for Hints

$$= \frac{y''x' - y'x''}{x'^3}$$

$$\therefore \quad \frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left(1+\left[\dfrac{dy}{dx}\right]^2\right)^{3/2}}$$

$$= \frac{y''x' - y'x''}{x'^3 \cdot \left[1+\left(\dfrac{y'}{x'}\right)^2\right]^{3/2}}$$

$$= \frac{x'y'' - y'x''}{\left(x'^2 + y'^2\right)^{3/2}}$$

Hence $\dfrac{1}{\rho} = \dfrac{x'y'' - y'x''}{\left(x'^2 + y'^2\right)^{3/2}}$ -- (3.34)

this proves the problem.

Example 3.3.4 :

Prove that the radius of curvature at any point of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ is $4a\cos\left(\theta/2\right)$.

Proof : Given that $x = a(\theta + \sin\theta)$ -------------------------------------- (3.35)

and $y = a(1 - \cos\theta)$ -- (3.36)

Differentiate (3.35) twice with respect to θ, we get,

$$\frac{dx}{d\theta} = a(1+\cos\theta) \text{ and } \frac{d^2x}{d\theta^2} = -a\sin\theta.$$

Again differentiate (3.36) twice with respect to θ, we get,

$$\frac{dy}{d\theta} = a\sin\theta \text{ and } \frac{d^2x}{d\theta^2} = a\cos\theta.$$

Space for Hints

Using (3.34), we have, $\frac{1}{\rho} = \frac{x'y'' - y'x''}{\left(x'^2 + y'^2\right)^{3/2}}$

(i.e) $$\frac{1}{\rho} = \frac{a(1+\cos\theta)\cdot a\cos\theta - a\sin\theta\cdot(-a\sin\theta)}{\left(a^2(1+\cos\theta)^2 + a^2\sin^2\theta\right)^{3/2}}.$$

$$= \frac{a^2(1+\cos\theta)}{a^3\left(2(1+\cos\theta)\right)^{3/2}}$$

$$= \frac{2\cos^2(\theta/2)}{a\left(4\cos^2(\theta/2)\right)^{3/2}}$$

$$= \frac{1}{4a\cos(\theta/2)}$$

Therefore $\rho = 4a\cos(\theta/2)$.

(i.e) the radius of curvature is $4a\cos(\theta/2)$.

This proves the problem.

Example 3.3.5 :

Prove that the radius of curvature at the point $(a\cos^3\theta, a\sin^3\theta)$ on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is $3a\sin\theta\cos\theta$.

Proof : Given that $x^{2/3} + y^{2/3} = a^{2/3}$ ---------------------------------- (3.37)

Let $x = a\cos^3\theta$ -- (3.38)

and $y = a\sin^3\theta$ -- (3.39)

Now differentiate (3.38) twice with respect to θ, we get,

$$\frac{dx}{d\theta} = 3a\cos^2\theta\cdot(-\sin\theta) = -3a\cos^2\theta\cdot\sin\theta$$

and $$\frac{d^2x}{d\theta^2} = -3a(\cos^2\theta\cdot\cos\theta + \sin\theta\cdot 2\cos\theta\cdot(-\sin\theta))$$

Space for Hints

$$= -3a(\cos^3\theta - 2\cdot\sin^2\theta\cdot\cos\theta)$$

Now differentiate (3.39) twice with respect to θ, we get,

$$\frac{dy}{d\theta} = a3\cdot\sin^2\theta\cdot\cos\theta$$

and $\dfrac{d^2y}{d\theta^2} = 3a(\sin^2\theta(-\sin\theta) + \cos\theta\cdot 2\sin\theta\cdot\cos\theta)$

$$= 3a\sin\theta\cdot(-\sin^2\theta + 2\cdot\cos^2\theta)$$

$$= 3a\sin\theta\cdot(2\cdot\cos^2\theta - \sin^2\theta)$$

Using (3.34), we have, $\dfrac{1}{\rho} = \dfrac{x'y'' - y'x''}{\left(x'^2 + y'^2\right)^{3/2}}$

$$= \frac{-9a^2\cos^2\theta\sin^2\theta(2\cos^2\theta - \sin^2\theta) + 9a^2\sin^2\theta\cdot\cos^2\theta(\cos^2\theta - 2\sin^2\theta)}{\left[\left(-3a\cos^2\theta\cdot\sin\theta\right)^2 + \left(3a\sin^2\theta\cdot\cos\theta\right)^2\right]^{3/2}}$$

$$= \frac{9a^2\cos^2\theta\sin^2\theta(-2\cos^2\theta + \sin^2\theta + \cos^2\theta - 2\sin^2\theta)}{\left[\left(9a^2\cos^4\theta\cdot\sin^2\theta\right) + \left(9a^2\sin^4\theta\cdot\cos^2\theta\right)\right]^{3/2}}$$

$$= \frac{9a^2\cos^2\theta\sin^2\theta(-\cos^2\theta - \sin^2\theta)}{\left[9a^2\cos^2\theta\cdot\sin^2\theta(\cos^2\theta + \sin^2\theta)\right]^{3/2}}$$

$$= \frac{-9a^2\cos^2\theta\sin^2\theta}{\left[9a^2\cos^2\theta\cdot\sin^2\theta\right]^{3/2}}$$

$$= \frac{-(3a\cos\theta\ \sin\theta)^2}{\left[3a\cos\theta\cdot\sin\theta\right]^3}$$

$$= \frac{-1}{3a\cos\theta\cdot\sin\theta}$$

$\therefore\ \ \rho = 3a\cos\theta\cdot\sin\theta$ (numerically)

This proves the problem.

Space for Hints

Check your progress

Question :

Find ρ at the point θ of the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$.

(Answer $\rho = a\theta$)

3.4. Centre of curvature

Definition : The centre of curvature at any point P of a curve is the point which lies on the positive direction of the normal at P and is a distance ρ, form it.

Example 3.4.1 :

Derive the coordinates of the centre of curvature for any point $P(x,y)$ of the curve $y = f(x)$.

Derivation : Let the positive direction of the tangent make angle ψ, with $x-$axiz so that the positive direction of the normal makes angle $\psi + \frac{\pi}{2}$ with the $x-$axiz.

Hence the equation of the normal at $P(x,y)$ is

$$\frac{X-x}{\cos\left(\psi + \frac{\pi}{2}\right)} = \frac{Y-y}{\sin\left(\psi + \frac{\pi}{2}\right)}$$

(i.e) $\dfrac{X-x}{-\sin\psi} = \dfrac{Y-y}{\cos\psi}$.

Thus the coordinates of the point on the normal at a distance ρ are $(x - \rho\sin\psi, y + \rho\cos\psi)$.

(i.e) $X = x - \rho\sin\psi$ and $Y = y + \rho\cos\psi$ ----------------------------- (3.40)

Space for Hints

We know that $\rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2}$, $\sin\psi = \frac{y_1}{\sqrt{1+y_1^2}}$ and $\cos\psi = \frac{1}{\sqrt{1+y_1^2}}$

Therefore (3.40) changes as $X = x - \frac{y_1(1+y_1^2)}{y_2}$ and $Y = y + \frac{1+y_1^2}{y_2}$

Example 3.4.2 :

Find the coordinates of the centre of curvature at a point $P(x, y)$ of the parabola $y^2 = 4ax$

Solution : Given that $y^2 = 4ax$ -- (3.41)

Differentiate (3.41) with respect to x, we get,

$$2y\frac{dy}{dx} = 4a$$

(i.e) $\frac{dy}{dx} = \frac{2a}{y}$

(i.e) $y_1 = \frac{2a}{y}$

and $y_2 = -\frac{2a}{y^2}\cdot\frac{dy}{dx}$

(i.e) $y_2 = -\frac{2a}{y^2}\cdot\frac{2a}{y}$

(i.e) $y_2 = -\frac{4a^2}{y^3}$

Let (X, Y) be the centre of curvature.

We know that

$$X = x - \frac{y_1(1+y_1^2)}{y_2}$$

Space for Hints

$$= x - \frac{\frac{2a}{y}\left(1+\left(\frac{2a}{y}\right)^2\right)}{-\frac{4a^2}{y^3}}$$

$$= x + \frac{2a}{y}\cdot\frac{y^3}{4a^2}\cdot\left(1+\frac{4a^2}{y^2}\right)$$

$$= x + \frac{1}{2a}\cdot y^2\cdot\left(\frac{y^2+4a^2}{y^2}\right)$$

$$= x + \frac{1}{2a}\cdot(4ax+4a^2)$$

$$= x+2x+2a$$

$$= 3x+2a$$

Thus $X = 3x+2a$

and $Y = y + \frac{1+y_1^2}{y_2}$

$$= y + \frac{1+\left(\frac{2a}{y}\right)^2}{-\frac{4a^2}{y^3}}$$

$$= y - \frac{y^3}{4a^2}\left[1+\left(\frac{4a^2}{y^2}\right)\right]$$

$$= y - \frac{y^3}{4a^2}\left[\frac{y^2+4a^2}{y^2}\right]$$

$$= y - \frac{y}{4a^2}\left[y^2+4a^2\right]$$

$$= \frac{4a^2y - y^3 - 4a^2y}{4a^2}$$

Space for Hints

$$= \frac{-y^3}{4a^2}$$

$$= \frac{\pm(4ax)^{3/2}}{4a^2}$$

$$= \frac{\pm 4ax\sqrt{4ax}}{4a^2}$$

$$= \frac{\pm x\sqrt{4ax}}{a}$$

$$= \pm x\sqrt{\frac{4ax}{a^2}}$$

$$= \pm 2x\sqrt{\frac{x}{a}}$$

Thus $Y = \pm 2x\sqrt{\frac{x}{a}}$

Hence the centre of curvature is $\left(3x+2a, \pm 2x\sqrt{\frac{x}{a}}\right)$

Example 3.4.3 :

Find the centre of curvature of the $x^{2/3} + y^{2/3} = a^{2/3}$.

Solution : Given that $x^{2/3} + y^{2/3} = a^{2/3}$ ------------------------------ (3.42)

We know that the parametric equations of (3.42) are $x = a\cos^3\theta$, $y = a\sin^3\theta$.

Now $x = a\cos^3\theta$

$\therefore\ \frac{dx}{d\theta} = a3\cos^2\theta(-\sin\theta)$

(i.e) $\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$

and $\frac{dy}{d\theta} = a3\sin^2\theta(\cos\theta)$

Space for Hints

(i.e) $\dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

$\therefore\ y_1 = \dfrac{dy}{dx}$

$= \dfrac{dy/d\theta}{dx/d\theta}$

$= \dfrac{-\cancel{3}\cancel{a}\cos^2\theta\sin\theta}{\cancel{3}\cancel{a}\sin^2\theta\cos\theta}$

$= -\tan\theta$

and $y_2 = \dfrac{d^2y}{dx^2}$

$= \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$

$= \dfrac{d}{d\theta}\left(\dfrac{dy}{dx}\right)\cdot\dfrac{d\theta}{dx}$

$= \dfrac{d}{d\theta}\left(-\tan\theta\right)\cdot\left(\dfrac{1}{-3a\cos^2\theta\sin\theta}\right)$

$= \sec^2\theta\cdot\dfrac{1}{3a\cos^2\theta\sin\theta}$

$= \dfrac{1}{3a}\sec^4\theta\operatorname{cosec}\theta$

Let (X,Y) be the required centre of curvature.

We know that $X = x - \dfrac{y_1(1+y_1^2)}{y_2}$ and $Y = y + \dfrac{1+y_1^2}{y_2}$

Now $X = x - \dfrac{y_1(1+y_1^2)}{y_2}$

$= a\cos^3\theta - \dfrac{(-\tan\theta)\cdot\left(1+\tan^2\theta\right)}{\dfrac{1}{3a}\sec^4\theta\operatorname{cosec}\theta}$

Space for Hints

$$= a\cos^3\theta + 3a\cdot\frac{\sin\theta}{\cos\theta}\cdot\frac{\sec^2\theta\cdot\sin\theta}{\sec^4\theta}$$

$$= a\cos^3\theta + 3a\cdot\sin^2\theta\cdot\cos\theta$$

and $Y = y + \dfrac{1+y_1^2}{y_2}$

$$= a\sin^3\theta + \frac{\left(1+\tan^2\theta\right)}{\frac{1}{3a}\sec^4\theta\,\text{cosec}\,\theta}$$

$$= a\sin^3\theta + 3a\cdot\frac{\sec^2\theta\cdot\sin\theta}{\sec^4\theta}$$

$$= a\sin^3\theta + 3a\cdot\sin\theta\cdot\cos^2\theta$$

Hence $\left(a\cos^3\theta + 3a\cdot\sin^2\theta\cdot\cos\theta,\ a\sin^3\theta + 3a\cdot\sin\theta\cdot\cos^2\theta\right)$, which is the required centre of curvature.

Check your progress

Questions :

(1) Find the coordinates of the centre of curvature of $y = x^2$ at $\left(\frac{1}{2}, \frac{1}{4}\right)$

(2) Find the coordinates of the centre of curvature of $xy = c^2$ at (c, c)

3.5. Evolute

Definition : The locus of centre of curvature for a curve is called the evolute of the curve.

Space for Hints

Example 3.5.1

Find the evolute of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Solution : Given that $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ -- (3.43)

We know that the parametric coordinates of the ellipse (3.43) are $(a\cos\theta, b\sin\theta)$.

(i.e) $x = a\cos\theta$ and $y = b\sin\theta$.

Thus $\frac{dx}{d\theta} = -a\sin\theta$ and $\frac{dy}{d\theta} = b\cos\theta$

$$\therefore \quad y_1 = \frac{dy}{dx}$$

$$= \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{b\cos\theta}{-a\sin\theta}$$

$$= -\frac{b}{a}\cdot\cot\theta$$

and $y_2 = \frac{d^2y}{dx^2}$

$$= \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\cdot\frac{d\theta}{dx}$$

$$= \frac{d}{d\theta}\left(-\frac{b}{a}\cdot\cot\theta\right)\cdot\frac{1}{(-a\sin\theta)}$$

$$= \frac{b}{a}\cdot\operatorname{cosec}^2\theta\cdot\frac{1}{(-a\sin\theta)}$$

$$= -\frac{b}{a^2}\cdot\operatorname{cosec}^3\theta$$

Space for Hints

Let (X, Y) be the centre of curvature.

We know that $X = x - \dfrac{y_1(1+y_1^2)}{y_2}$ and $Y = y + \dfrac{1+y_1^2}{y_2}$

Now $X = x - \dfrac{y_1(1+y_1^2)}{y_2}$

$$= a\cos\theta - \frac{-\frac{b}{a}\cdot\cot\theta\left(1+\left(-\frac{b}{a}\cdot\cot\theta\right)^2\right)}{-\frac{b}{a^2}\cdot\cos ec^3\theta}$$

$$= a\cos\theta - a\cdot\frac{\cos\theta}{\sin\theta}\left(1+\left(\frac{b^2}{a^2}\cdot\cot^2\theta\right)\right)\cdot\sin^3\theta$$

$$= a\cos\theta - a\cdot\cos\theta\cdot\sin^2\theta\cdot\left(1+\left(\frac{b^2}{a^2}\cdot\frac{\cos^2\theta}{\sin^2\theta}\right)\right)$$

$$= a\cos\theta - a\cdot\cos\theta\cdot\sin^2\theta\cdot\left(\frac{a^2\sin^2\theta+b^2\cos^2\theta}{a^2\sin^2\theta}\right)$$

$$= a\cos\theta - \cos\theta\cdot\frac{1}{a}\left(a^2\sin^2\theta+b^2\cos^2\theta\right)$$

$$= \frac{a^2\cos\theta - \cos\theta\cdot\left(a^2\sin^2\theta+b^2\cos^2\theta\right)}{a}$$

$$= \frac{1}{a}\cos\theta\left[a^2 - a^2\sin^2\theta - b^2\cos^2\theta\right]$$

$$= \frac{1}{a}\cos\theta\left[a^2(1-\sin^2\theta) - b^2\cos^2\theta\right]$$

$$= \frac{1}{a}\cos\theta\left[a^2\cos^2\theta - b^2\cos^2\theta\right]$$

$$= \frac{1}{a}(a^2-b^2)\cos^3\theta$$

(i.e) $X = \dfrac{1}{a}(a^2-b^2)\cos^3\theta$

Space for Hints

$$\therefore \cos^3\theta = \frac{aX}{a^2-b^2}$$

$$\text{Thus } \cos\theta = \left(\frac{aX}{a^2-b^2}\right)^{1/3} \text{-- (3.44)}$$

$$\text{Again } Y = y + \frac{1+y_1^2}{y_2}$$

$$= b\sin\theta + \frac{1+\dfrac{b^2}{a^2}\cot^2\theta}{-\dfrac{b}{a^2}\operatorname{cosec}^3\theta}$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta \cdot \left[1+\frac{b^2\cos^2\theta}{a^2\sin^2\theta}\right]$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta \cdot \left[\frac{a^2\sin^2\theta + b^2\cos^2\theta}{a^2\sin^2\theta}\right]$$

$$= b\sin\theta - \frac{1}{b}\sin\theta \cdot \left[a^2\sin^2\theta + b^2\cos^2\theta\right]$$

$$= \frac{1}{b}\cdot\left(b^2\sin\theta - \sin\theta\cdot\left[a^2\sin^2\theta + b^2\cos^2\theta\right]\right)$$

$$= \frac{1}{b}\cdot\sin\theta\cdot\left(b^2 - a^2\sin^2\theta - b^2\cos^2\theta\right)$$

$$= \frac{1}{b}\cdot\sin\theta\cdot\left(b^2\sin^2\theta - a^2\sin^2\theta\right)$$

$$= \frac{1}{b}\cdot\sin^3\theta\cdot\left(b^2-a^2\right)$$

$$= -\frac{a^2-b^2}{b}\sin^3\theta$$

$$\text{(i.e) } Y = -\frac{a^2-b^2}{b}\sin^3\theta$$

$$\therefore \sin^3\theta = \frac{-bY}{a^2-b^2}$$

Space for Hints

Thus $\sin\theta = \left(\dfrac{-bY}{a^2-b^2}\right)^{1/3}$ -- (3.45)

Squaring (3.44), (3.45) and then adding, we get,

$$\left(\frac{aX}{a^2-b^2}\right)^{2/3} + \left(\frac{-bY}{a^2-b^2}\right)^{2/3} = 1$$

(i.e) $\left(\dfrac{aX}{a^2-b^2}\right)^{2/3} + \left(\dfrac{bY}{a^2-b^2}\right)^{2/3} = 1$

(i.e) $(aX)^{2/3} + (bY)^{2/3} = (a^2-b^2)^{2/3}$

Therefore the locus of (X,Y) is $(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}$.

Which is the required evolute.

Example 3.5.2

Find the evolute of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$

Solution : Given that $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$

Now $\dfrac{dx}{d\theta} = a(1-\cos\theta)$ and $\dfrac{dy}{d\theta} = a\sin\theta$

$$\therefore \quad y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{\cancel{a}\sin\theta}{\cancel{a}(1-\cos\theta)}$$

$$= \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)}$$

$$= \cot(\theta/2)$$

Again $y_2 = \dfrac{d^2y}{dx^2}$

$$= \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Space for Hints

$$= \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\cdot\frac{d\theta}{dx}$$

$$= \frac{d}{d\theta}\left(\cot(\theta/2)\right)\cdot\frac{1}{a(1-\cos\theta)}$$

$$= -\frac{1}{2}\cdot\operatorname{cosec}^2\theta\frac{1}{2a\sin^2\theta}$$

$$= -\frac{1}{4a\sin^4\theta}$$

Let (X,Y) be the centre of curvature.

We know that $X = x-\frac{y_1(1+y_1^2)}{y_2}$ and $Y = y+\frac{1+y_1^2}{y_2}$

Now $X = x-\frac{y_1(1+y_1^2)}{y_2}$

$$= a(\theta-\sin\theta)+\frac{\cot(\theta/2)\left(1+\cot^2(\theta/2)\right)}{1/4\sin^4(\theta/2)}$$

$$= a(\theta-\sin\theta)+4\sin^4(\theta/2)\cdot\cot(\theta/2).\operatorname{cosec}^2(\theta/2)$$

$$= a(\theta-\sin\theta)+4\sin^2(\theta/2)\cdot\frac{\cos(\theta/2)}{\sin(\theta/2)}$$

$$= a(\theta-\sin\theta)+4\sin(\theta/2)\cdot\cos(\theta/2)$$

$$= a(\theta-\sin\theta)+2a\sin\theta$$

$$= a(\theta+\sin\theta)$$

(i.e) $X = a(\theta+\sin\theta)$

and $Y = y+\frac{1+y_1^2}{y_2}$

$$= a(1-\cos\theta)+\frac{1+\cot^2(\theta/2)}{-1/4\sin^4(\theta/2)}$$

$$= a(1-\cos\theta)-4a\sin^4(\theta/2)\operatorname{cosec}^2(\theta/2)$$

$$= a(1-\cos\theta)-2a\cdot 2\sin^2(\theta/2)$$

Space for Hints

$$= a(1-\cos\theta) - 2a\cdot(1-\cos\theta)$$

$$= -a(1-\cos\theta)$$

(i.e) $Y = -a(1-\cos\theta)$

Hence the locus of (X,Y) is $x = a(\theta+\sin\theta)$, $y = -a(1-\cos\theta)$ which is a equation of a cycloid.

Example 3.5.3 :

Find the evolute of $xy = c^2$

Solution : Given that $xy = c^2$

$$\therefore \quad y = \frac{c^2}{x}$$

Thus $y_1 = -\frac{c^2}{x^2}$ and $y_2 = \frac{2c^2}{x^3}$

Let (X,Y) be the centre of curvature.

We know that $X = x - \frac{y_1(1+y_1^2)}{y_2}$ and $Y = y + \frac{1+y_1^2}{y_2}$

Now $X = x - \frac{y_1(1+y_1^2)}{y_2}$

$$= x - \frac{-\frac{c^2}{x^2}\left(1+\frac{c^4}{x^4}\right)}{\frac{2c^2}{x^3}}$$

$$= x + \frac{x^3}{2c^2}\cdot\frac{c^2}{x^2}\cdot\left(\frac{x^4+c^4}{x^4}\right)$$

$$= x + \frac{1}{2x^3}\cdot(x^4+c^4)$$

$$= \frac{2x^4+x^4+c^4}{2x^3}$$

Space for Hints

$$= \frac{3x^4 + \left(c^2\right)^2}{2x^3}$$

$$= \frac{3x^4 + x^2y^2}{2x^3}$$

$$= \frac{3x^2 + y^2}{2x}$$

(i.e) $X = \dfrac{3x^2 + y^2}{2x}$ -- (3.46)

and $Y = y + \dfrac{1 + y_1^2}{y_2}$

$$= y + \frac{1 + \frac{c^4}{x^4}}{\frac{2c^2}{x^3}}$$

$$= y + \frac{x^3}{2c^2} \cdot \frac{x^4 + c^4}{x^4}$$

$$= y + \frac{1}{2c^2} \cdot \frac{x^4 + x^2y^2}{x}$$

$$= y + \frac{1}{2y} \cdot (x^2 + y^2)$$

$$= \frac{2y^2 + x^2 + y^2}{2y}$$

$$= \frac{3y^2 + x^2}{2y}$$

(i.e) $Y = \dfrac{3y^2 + x^2}{2y}$ -- (3.47)

Now (3.46) + (3.47) $\Rightarrow$ $X + Y = \dfrac{3x^2 + y^2}{2x} + \dfrac{3y^2 + x^2}{2y}$

Space for Hints

$$= \frac{y(3x^2 + y^2) + x(3y^2 + x^2)}{2xy}$$

$$= \frac{3x^2y + y^3 + 3xy^2 + x^3}{2xy}$$

$$= \frac{(x+y)^3}{2xy}$$

$$= \frac{(x+y)^3}{2c^2}$$

(i.e) $X + Y = \dfrac{(x+y)^3}{2c^2}$

Thus $(X+Y)^{2/3} = \dfrac{(x+y)^2}{\left(2c^2\right)^{2/3}}$ -- (3.48)

and $(3.46) - (3.47) \Rightarrow X - Y = \dfrac{3x^2 + y^2}{2x} - \dfrac{3y^2 + x^2}{2y}$

$$= \frac{y(3x^2 + y^2) - x(3y^2 + x^2)}{2xy}$$

$$= \frac{3x^2y + y^3 - 3xy^2 - x^3}{2xy}$$

$$= -\frac{(x-y)^3}{2xy}$$

$$= -\frac{(x-y)^3}{2c^2}$$

(i.e) $X - Y = -\dfrac{(x-y)^3}{2c^2}$

Thus $(X-Y)^{2/3} = \dfrac{(x-y)^2}{\left(2c^2\right)^{2/3}}$ -- (3.49)

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Now (3.48) – (3.49) $\Rightarrow (X+Y)^{2/3}-(X-Y)^{2/3}=\dfrac{(x+y)^2}{\left(2c^2\right)^{2/3}}-\dfrac{(x-y)^2}{\left(2c^2\right)^{2/3}}$

$$=\frac{1}{\left(2c^2\right)^{2/3}}\cdot\left((x+y)^2-(x-y)^2\right)$$

$$=\frac{4xy}{\left(2c^2\right)^{2/3}}$$

$$=\left(2c\right)^{2/3}$$

(i.e) $(X+Y)^{2/3}-(X-Y)^{2/3}=\left(2c\right)^{2/3}$

Hence the locus of (X,Y) is $\left(x+y\right)^{2/3}-\left(x-y\right)^{2/3}=\left(2c\right)^{2/3}$

Check your progress

Question

Show that the evolute of the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ is $(ax)^{2/3}+(by)^{2/3}=(a^2+b^2)^{2/3}$

Summary

In this unit we have leaned how to find envelopes, curvatures, radius of curvature, centre of curvature and evolutes.

Further Reading

You can also refer the following books for further reading.

(1) Calculus by Arumugam and Isaac

(2) Differential Calculus by Shanti Narayanan

Space for Hints

UNIT IV

RADIUS OF CURVATURES in POLAR COORDINATES AND INTEGRAL CALCULUS

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Objectives :

In this unit we are going to discuss radius of curvature, pedal equation and then we discuss definite integrals and properties of integration.

After completing this unit, students may able to know

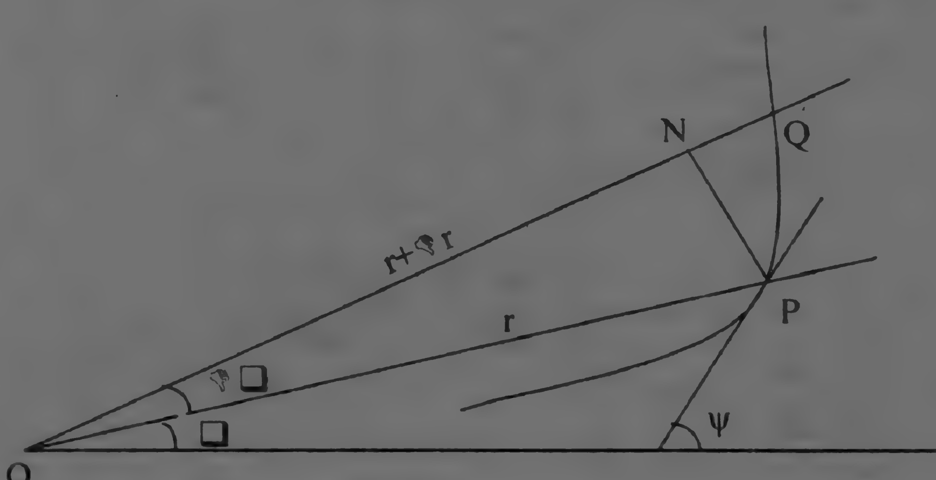
- Radius of curvature in polar coordinates
- Pedal equations
- Definite integrals
- Properties of integrals

Introduction

In continuation of previous unit, in this unit we shall discuss radius of curvature in polar coordinates and pedal equation of various curves.

4.1. Radius of Curvature in polar coordinates

Now we shall find the radius of curvature in polar coordinates.



Let $r = f(\theta)$ be the equation of the in the polar form.

From the figure, it easy to see that $\psi = \theta + \phi$

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$$\therefore \frac{d\psi}{d\theta} = 1 + \frac{d\phi}{d\theta} \quad \text{------------------------------------(4.1)}$$

Already we know that $tan\phi = r\dfrac{d\theta}{dr}$

(i.e) $tan\phi = r\dfrac{1}{\dfrac{dr}{d\theta}}$

Differentiate with respect to θ we get,

$$\sec^2\theta \frac{d\phi}{d\theta} = \frac{\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{\left(\dfrac{dr}{d\theta}\right)^2}$$

$$\text{Thus } \frac{d\phi}{d\theta} = \frac{1}{\sec^2\theta}\cdot\frac{\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{\left(\dfrac{dr}{d\theta}\right)^2}$$

$$= \frac{1}{1+\tan^2\theta}\cdot\frac{\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{\left(\dfrac{dr}{d\theta}\right)^2}$$

$$= \frac{1}{1+\dfrac{r^2}{\left(\dfrac{dr}{d\theta}\right)^2}}\cdot\frac{\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{\left(\dfrac{dr}{d\theta}\right)^2}$$

$$= \frac{1}{\left[\left(\dfrac{dr}{d\theta}\right)^2 + r^2\right]}\cdot\left[\left(\dfrac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right]$$

Thus (4.1) $\Rightarrow \dfrac{d\psi}{d\theta} = 1 + \dfrac{1}{\left[\left(\dfrac{dr}{d\theta}\right)^2 + r^2\right]} \cdot \left[\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}\right]$

$$= \frac{\left[\left(\dfrac{dr}{d\theta}\right)^2 + r^2\right] + \left[\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}\right]}{\left[\left(\dfrac{dr}{d\theta}\right)^2 + r^2\right]}$$

$$= \frac{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{\left[\left(\dfrac{dr}{d\theta}\right)^2 + r^2\right]} \quad \text{---} \ (4.2)$$

Also we know that $\dfrac{ds}{d\theta} = \left(r^2 + \left[\dfrac{dr}{d\theta}\right]^2\right)^{1/2}$ -------------------------- (4.3)

Hence $\rho = \dfrac{ds}{d\psi}$

$$= \frac{ds}{d\theta} \cdot \frac{d\theta}{d\psi}$$

$$= \left(r^2 + \left[\frac{dr}{d\theta}\right]^2\right)^{1/2} \cdot \left\{\frac{\left[\left(\dfrac{dr}{d\theta}\right)^2 + r^2\right]}{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}\right\}$$

$$= \frac{\left(r^2 + \left[\dfrac{dr}{d\theta}\right]^2\right)^{3/2}}{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}$$

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Space for Hints

$$\boxed{\text{(i.e)}\ \rho = \frac{\left(r^2 + \left[\frac{dr}{d\theta}\right]^2\right)^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}}$$

Example 4.1.1 :

Find the radius of curvature of the curve $r = a(1-\cos\theta)$.

Solution : Given that $r = a(1-\cos\theta)$ ------------------------------------ (4.4)

Differentiate (4.4) twice with respect to θ, we get,

$$\frac{dr}{d\theta} = a\sin\theta$$

and $\dfrac{d^2r}{d\theta^2} = a\cos\theta$

We know that $\rho = \dfrac{\left(r^2 + \left[\frac{dr}{d\theta}\right]^2\right)^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$

$$\text{(i.e)}\ \rho = \frac{\left[a^2(1-\cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}}{a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a\cos\theta\, a(1-\cos\theta)}$$

$$= \frac{a^3\left[1-2\cos\theta+\cos^2\theta+\sin^2\theta\right]^{3/2}}{a^2\left[1-2\cos\theta+\cos^2\theta+2\sin^2\theta-\cos\theta+\cos^2\theta\right]}$$

$$= \frac{a\left[1-2\cos\theta+1\right]^{3/2}}{\left[1-3\cos\theta+2\right]}$$

$$= \frac{a(2)^{2/3}\left[1-\cos\theta\right]^{3/2}}{3\left[1-\cos\theta\right]}$$

$$= \frac{a2\sqrt{2}\left[1-\cos\theta\right]\sqrt{1-\cos\theta}}{3\left[1-\cos\theta\right]}$$

$$= \frac{a2\sqrt{2}}{3}\sqrt{\frac{r}{a}}$$

$$= \frac{2}{3}\sqrt{2ar}$$

Hence the radius of curvature of the curve is $\frac{2}{3}\sqrt{2ar}$.

Example 4.1.2 :

Show that the radius of curvature at any point of the cardioid $r = a(1+\cos\theta)$ is $\frac{2}{3}\sqrt{2ar}$ and prove that $\frac{\rho^2}{r}$ is constant.

Proof : Given that $r = a(1+\cos\theta)$ -- (4.5)

Differentiate (4.5) twice with respect to θ, we get,

$$r_1 = \frac{dr}{d\theta} = -a\sin\theta$$

and $$r_2 = \frac{d^2r}{d\theta^2} = -a\cos\theta$$

We know that $$\rho = \frac{\left(r^2 + \left[\frac{dr}{d\theta}\right]^2\right)^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

$$= \frac{\left[a^2(1+\cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}}{a^2(1+\cos\theta)^2 - a(1+\cos\theta)(-a\cos\theta) + 2a^2\sin^2\theta}$$

$$= \frac{a^3\left[1+2\cos\theta+\cos^2\theta+\sin^2\theta\right]^{3/2}}{a^2\left[1+2\cos\theta+\cos^2\theta+\cos\theta+\cos^2\theta+2\sin^2\theta\right]}$$

$$= \frac{a\left[1+2\cos\theta+1\right]^{3/2}}{\left[1+3\cos\theta+2\right]}$$

Space for Hints

Space for Hints

$$= \frac{a(2)^{2/3}\left[1+\cos\theta\right]^{3/2}}{3\left[1+\cos\theta\right]}$$

$$= \frac{a2\sqrt{2}\left[1+\cos\theta\right]\sqrt{1+\cos\theta}}{3\left[1+\cos\theta\right]}$$

$$= \frac{a2\sqrt{2}}{3}\sqrt{\frac{r}{a}}$$

$$= \frac{2}{3}\sqrt{2ar}$$

Hence the radius of curvature of the curve is $\frac{2}{3}\sqrt{2ar}$.

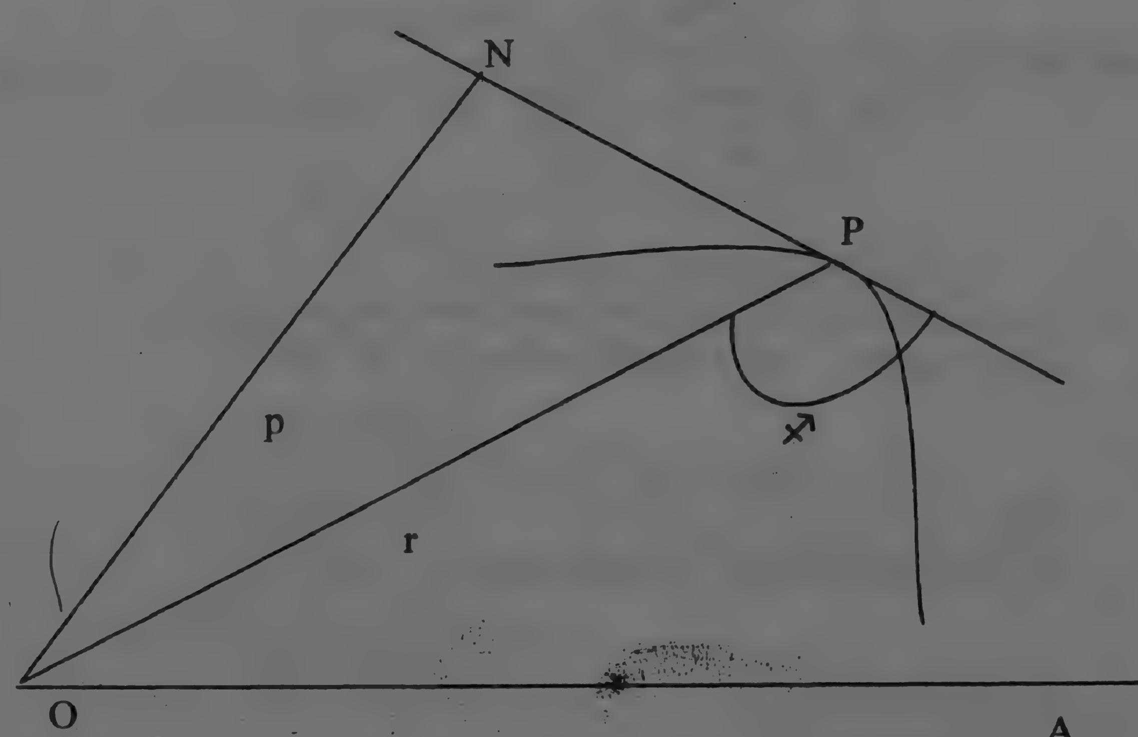
Again $\frac{\rho^2}{r} = \frac{1}{r}\cdot\frac{4}{9}\cdot(2ar)$

$$= \frac{8a}{9} \text{ which is constant.}$$

4.2. Radius of Curvature in polar coordinates

Some times we write the equation of the curve in pedal equation form and it is called p-r equation of the curve.

Now we shall derive p-r equation of a curve.



Let O be the pole and OA be the initial line.

Let P be a point on the curve $p = f(r)$.

Draw ON perpendicular to the tangent at P.

Let $ON = p$

Now from the ΔOPN,

$$ON = p$$
$$= r\sin(O\hat{P}N)$$
$$= r\sin(180-\phi)$$
$$= r\sin\phi$$

(i.e) $p = r\sin\phi$

Thus $$\frac{1}{p^2} = \frac{1}{r^2\sin^2\phi}$$

$$= \frac{1}{r^2}\cdot\cos ec^2\phi$$

$$= \frac{1}{r^2}\cdot(1+\cot^2\phi)$$

$$= \frac{1}{r^2}\cdot\left(1+\left[\frac{1}{r}\left(\frac{dr}{d\theta}\right)\right]^2\right)$$

$$= \frac{1}{r^2}\cdot\left(1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right)$$

(i.e) $$\frac{1}{p^2} = \frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \quad \text{--} \quad (4.6)$$

which is the required p-r equation of the curve.

Note (1) :

Let $r = \frac{1}{u}$

$$\therefore \frac{dr}{d\theta} = -\frac{1}{u^2}\cdot\frac{du}{d\theta}$$

Thus (4.6) changes as $\frac{1}{p^2} = u^2+u^4\left(-\frac{1}{u^2}\cdot\frac{du}{d\theta}\right)^2$

Space for Hints

Space for Hints

(i.e) $\dfrac{1}{p^2} = u^2 + u^4 \cdot \dfrac{1}{u^4} \cdot \left(\dfrac{du}{d\theta}\right)^2$

(i.e) $\dfrac{1}{p^2} = u^2 + \left(\dfrac{du}{d\theta}\right)^2$

The above equation is called differential equation of the curve in p-r form.

Note (2) : We know that $\dfrac{ds}{d\theta} = \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}$

and $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$

$$= \frac{1}{r^4}\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]$$

$\therefore\ \dfrac{1}{p} = \dfrac{1}{r^2}\sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}$

(i.e) $\dfrac{1}{p} = \dfrac{1}{r^2}\dfrac{ds}{d\theta}$

(i.e) $\dfrac{ds}{d\theta} = \dfrac{r^2}{p}$

Note (3) :

We know that $\sin\phi = r\dfrac{d\theta}{ds}$ -- (4.7)

$\cos\phi = \dfrac{dr}{ds}$ -- (4.8)

$\tan\phi = r\dfrac{d\theta}{dr}$ -- (4.9)

and $p = r\sin\phi$ -- (4.10)

Now differentiate (4.10) with respect to r, we get,

$$\frac{dp}{dr} = \sin\phi + r\cos\phi \frac{d\phi}{dr}$$

$$= r\frac{d\theta}{ds} + r\frac{dr}{ds}\frac{d\phi}{dr} \text{ (using (4.7) and (4.8))}$$

$$= r\frac{d\theta}{ds} + r\frac{d\phi}{ds}$$

$$= r\left(\frac{d\theta}{ds} + \frac{d\phi}{ds}\right)$$

$$= r\frac{d}{ds}(\theta + \phi)$$

$$= r\frac{d\psi}{ds}$$

$$\therefore \quad \rho = r\frac{ds}{d\psi}$$

(i.e) $\rho = r\dfrac{dr}{dp}$

From the above formula, we can easily find radius of curvature of a curve, which is given, in the polar form.

Example 4.2.1 :

Find the radius of curvature at (p,r) on the ellipse $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} - \dfrac{r^2}{a^2b^2}$.

Solution : Given that $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} - \dfrac{r^2}{a^2b^2}$ ------------------------ (4.11)

Differentiate with respect to r, we get,

$$-\frac{1}{p^3}\frac{dp}{dr} = -\frac{2r}{a^2b^2}$$

(i.e) $\dfrac{dp}{dr} = \dfrac{rp^3}{a^2b^2}$

We know that $\rho = r\dfrac{dr}{dp}$

(i.e) $\rho = r \cdot \dfrac{a^2b^2}{rp^3}$

Space for Hints

Space for Hints

(i.e) $\rho = \dfrac{a^2b^2}{p^3}$

which is the required radius of curvature of (4.11)

Example 4.2.2 :

Find the radius of curvature at any point of the curve $r^n = a^n \cos n\theta$

Proof : Given that $r^n = a^n \cos n\theta$ -- (4.12)

Taking log on both sides, we get,

$n \log r = n \log a + \log \cos n\theta$ -- (4.13)

Differentiate (4.13) with respect to θ , we get,

$$\frac{n}{r} \cdot \frac{dr}{d\theta} = \frac{1}{\cos n\theta} \cdot (-a \sin n\theta)$$

(i.e) $\dfrac{dr}{d\theta} = -r \tan n\theta$

We know that $\tan \phi = r \dfrac{d\theta}{dr}$

$$= r \cdot \left(\frac{1}{-r \tan n\theta} \right)$$

$$= -\cot n\theta$$

$$= \tan \left(\frac{\pi}{2} + n\theta \right)$$

$\therefore \quad \phi = \dfrac{\pi}{2} + n\theta$ --- (4.14)

Again $p = r \sin \phi$

$$= r \sin \left(\frac{\pi}{2} + n\theta \right)$$

$$= r \cos n\theta$$

$$= r \cdot \frac{r^n}{a^n}$$

$$= \frac{r^{n+1}}{a^n}$$

(i.e) $a^n p = r^{n+1}$ -- (4.15)

Differentiate (4.15) with respect to r, we get,

$$a^n \frac{dp}{dr} = (n+1)r^n.$$

Again we know that $\rho = r\frac{dr}{dp}$

(i.e) $\rho = r \cdot \dfrac{a^n}{(n+1)r^n}$

(i.e) $\rho = \dfrac{a^n}{(n+1)r^{n-1}}$

which radius of curvature of (4.12).

Example 4.2.3 :

Find the radius of curvature at any point of the curve $r^2 \cos 2\theta = a^2$

Proof : Given that $r^2 \cos 2\theta = a^2$ -------------------------------------- (4.16)

Taking log on both sides, we get,

$2\log r + \log\cos 2\theta = 2\log a$ -- (4.17)

Differentiate (4.17) with respect to θ , we get,

$$\frac{2}{r}\cdot\frac{dr}{d\theta} + \frac{1}{\cos 2\theta}\cdot(-2\sin 2\theta) = 0$$

(i.e) $\dfrac{dr}{d\theta} = r\tan 2\theta$

We know that $\tan\phi = r\dfrac{d\theta}{dr}$

$$= r\cdot\left(\frac{1}{r\tan 2\theta}\right)$$

$$= \cot 2\theta$$

$$= \tan\left(\frac{\pi}{2} - 2\theta\right)$$

$\therefore\ \phi = \dfrac{\pi}{2} - n\theta$ -- (4.18)

Again $p = r\sin\phi$

Space for Hints

Space for Hints

$$= r\sin\left(\frac{\pi}{2}-2\theta\right)$$

$$= r\cos 2\theta$$

$$= r\cdot\frac{a^2}{r^2}$$

$$= \frac{a^2}{r} \quad \text{---(4.19)}$$

Differentiate (4.19) with respect to r, we get,

$$\frac{dp}{dr} = -\frac{a^2}{r^2}.$$

Again we know that $\rho = r\dfrac{dr}{dp}$

(i.e) $\rho = r\cdot\left(-\dfrac{r^2}{a^2}\right)$

(i.e) $\rho = -\dfrac{r^3}{a^2}$

(i.e) $\rho = \dfrac{r^3}{a^2}$ (numerically)

which radius of curvature of (4.16).

Example 4.2.4 :

Find $p-r$ equation of $r = \dfrac{a}{2}(1-\cos\theta)$

Solution : Given that $r = \dfrac{a}{2}(1-\cos\theta)$ ---------------------------------- (4.20)

$$\therefore \frac{dr}{d\theta} = \frac{a}{2}\sin\theta$$

We know that the formula for $p-r$ equation is $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$

(i.e) $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\sin^2\theta$

Space for Hints

$$= \frac{4r^2 + a^2\sin^2\theta}{4r^4}$$

$$= \frac{(2r)^2 + a^2\sin^2\theta}{4r^4}$$

$$= \frac{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}{4r^4}$$

$$= \frac{a^2(1-2\cos\theta+\cos^2\theta+\sin^2\theta)}{4r^4}$$

$$= \frac{a^2(1-2\cos\theta+1)}{4r^4}$$

$$= \frac{2a^2(1-\cos\theta)}{4r^4}$$

$$= \frac{2a^2 \cdot \frac{2r}{a}}{4r^4}$$

$$= \frac{a}{r^3}$$

(i.e) $\frac{1}{p^2} = \frac{a}{r^3}$

Hence $ap^2 = r^3$

which is the required $p-r$ equation of (4.20).

Example 4.2.5 :

Find $p-r$ equation of the curve $x^2 + y^2 = 2ax$ and hence find the radius of curvature.

Solution : Given that $x^2 + y^2 = 2ax$ ---------------------------------- (4.21)

put $x = r\cos\theta$ and $y = r\sin\theta$ in (4.21), we get,

$r^2\cos^2\theta + r^2\sin^2\theta = 2ar\cos\theta$

(i.e) $r^2 = 2ar\cos\theta$

(i.e) $r = 2a\cos\theta$ -- (4.22)

Space for Hints

Differentiate (4.22) with respect to θ, we have,

$$\frac{dr}{d\theta} = -2a\sin\theta$$

We know that the $p-r$ equation as $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

(i.e) $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} 4a^2 \sin^2\theta$

$$= \frac{r^2 + 4a^2\sin^2\theta}{r^4}$$

$$= \frac{4a^2\cos^2\theta + 4a^2\sin^2\theta}{r^4}$$

$$= \frac{4a^2}{r^4}$$

(i.e) $\frac{1}{p^2} = \frac{4a^2}{r^4}$.

$\therefore \quad p^2 = \frac{r^4}{4a^2}$

Thus $p = \frac{r^2}{2a}$ -- (4.23)

which is the required $p-r$ equation of (4.21)

Now we shall find radius of curvature of (4.21)

For that differentiate (4.23) with respect to r, we get,

$$\frac{dp}{dr} = \frac{r}{a}$$

Hence $\rho = r\frac{dp}{dr}$

(i.e) $\quad \rho = r \cdot \frac{a}{r}$

(i.e) $\quad \rho = a$ which the radius of curvature of (4.21)

Check your progress

Questions :

(1) Find the radius of curvature at (p,r) on the parabola $p^2 = ar$.

(2) Find the radius of curvature at (p,r) on the cardioid $2ap^2 = r^3$.

(3) Find the $p-r$ equation of $r = a\sin\theta$

(4) Find the $p-r$ equation of $r^n = a^n \sin n\theta$

(5) Prove that the pedal equation of $r = a^{\theta}$ is $p = kr$ where k is a constant.

Space for Hints

4.3. Definite integral

Upto the previous units and previous sections we discussed the differentiation and its applications. Now we shall study the process of integration.

Definition : The reverse process of *differentiation* is called *integration.*
We use the following standard results of integration that we shall use to solve the problems.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \text{ if } n \neq -1$$

$$\int x^{-1} dx = \int \frac{1}{x} dx = \log x + c$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax + c$$

$$\int \cos ax dx = \frac{1}{a}\sin ax + c$$

$$\int \sec^2 x \, dx = \tan x + c$$

$$\int \operatorname{cosec}^2 x \, dx = -\cot x + c$$

$$\int \sec x \tan x \, dx = \sec x + c$$

$$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$$

Space for Hints

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$$

$$\int a^x\,dx = \frac{a^x}{\log a} + c$$

$$\int \tan x\,dx = \log\sec x + c$$

$$\int \cot x\,dx = \log\sin x + c$$

$$\int \sec x\,dx = \log(\sec x + \tan x) + c = \log\left(\frac{\pi}{2} + \frac{x}{2}\right) + c$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{a^2 + x^2}}\,dx = \log\left(x + \sqrt{a^2 + x^2}\right) + c = \sinh^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \log\left(x + \sqrt{x^2 + a^2}\right) + c = \cosh^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\log\left(\frac{a + x}{a - x}\right) + c$$

$$\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\log\left(\frac{x - a}{x + a}\right) + c$$

$$\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \sqrt{a^2 + x^2}\,dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\log\left(x + \sqrt{a^2 + x^2}\right) + c$$

$$\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} + \frac{a^2}{2}\log\left(x + \sqrt{x^2 - a^2}\right) + c$$

$$\int \sinh x\,dx = \cosh x + c$$

$$\int \cosh x\,dx = \sinh x + c$$

Space for Hints

Note :

(1) $\int f(x)\,dx$ is called an indefinite integral and $\int_a^b f(x)\,dx$ is called definite integral.

(2) If $\int f(x)\,dx = F(x)+c$ then $\int_a^b f(x)\,dx = F(b)-F(a)$.

Example 4.3.1 :

Evaluate $\int_0^{10} (x^2-x+1)\,dx$

Solution : Let I = $\int_0^{10} (x^2-x+1)\,dx$

$$\therefore \quad I = \left[\frac{x^3}{3}-\frac{x^2}{2}+x\right]_0^{10}$$

$$= \left[\frac{10^3}{3}-\frac{10^2}{2}+10\right]-\left[0\right]$$

$$= \frac{1000}{3}-\frac{100}{2}+10$$

$$= \frac{880}{3}$$

Thus $\int_0^{10} (x^2-x+1)\,dx = \frac{880}{3}$

Example 4.3.2 :

Evaluate $\int_0^{\pi/4} \sin x \cos x\,dx$

Solution : Let I = $\int_0^{\pi/4} \sin x \cos x\,dx$

(i.e) $I = \frac{1}{2}\int_0^{\pi/4} \sin 2x\,dx$

Space for Hints

$$= \frac{1}{2}\left[-\frac{1}{2}\cos 2x\right]_0^{\pi/4}$$

$$= -\frac{1}{4}\left[\cos(\pi/2)-\cos(0)\right]$$

$$= -\frac{1}{4}\left[0-1\right]$$

$$= \frac{1}{4}$$

Thus $\int_0^{\pi/4} \sin x \cos x \, dx = \frac{1}{4}$.

Example 4.3.3 :

Evaluate $\int_0^2 \frac{5x}{x^2+4} dx$

Solution : Let I = $\int_0^2 \frac{5x}{x^2+4} dx$

Let $Nr. = A \cdot \frac{d}{dx}(Dr.) + B$

(i.e) $5x = A \cdot \frac{d}{dx}(x^2+4) + B$ -- (4.24)

(i.e) $5x = A \cdot (2x) + B$

Equating like coefficients on both sides, we get,

$2A = 5 \Rightarrow A = \frac{5}{2}$

and $B = 0$

$\therefore$ (4.24) $\Rightarrow$ $5x = \frac{5}{2} \cdot \frac{d}{dx}(x^2+4) + 0$

$\Rightarrow 5x = \frac{5}{2} \cdot \frac{d}{dx}(x^2+4)$

Thus I = $\int_0^2 \frac{\frac{5}{2}(2x)}{x^2+4} dx$

Space for Hints

$$= \frac{5}{2}\cdot\left[\log(x^2+4)\right]_0^2$$

$$= \frac{5}{2}\cdot\left[\log(4+4)-\log 4\right]$$

$$= \frac{5}{2}\cdot\log 2$$

Hence $\int_0^2 \frac{5x}{x^2+4}dx = \frac{5}{2}\cdot\log 2$.

Example 4.3.4 :

Evaluate $\int_0^1 \frac{15x^3}{\sqrt{1-x^8}}dx$

Solution : Let I = $\int_0^1 \frac{15x^3}{\sqrt{1-x^8}}dx$

Let $u = x^4$

$\therefore\ du = 4x^3\,dx$

$\Rightarrow\ x^3\,dx = \frac{1}{4}du$

When $x=0$ then $u=0$

When $x=1$ then $u=1$

$$\therefore\ \mathrm{I} = \frac{15}{4}\int_0^1 \frac{du}{\sqrt{1-u^2}}$$

$$= \frac{15}{4}\left[\sin^{-1}u\right]_0^1$$

$$= \frac{15}{4}\left[\sin^{-1}(1)-\sin^{-1}(0)\right]$$

$$= \frac{15}{4}\left[\frac{\pi}{2}-0\right]$$

$$= \frac{15\pi}{8}$$

Hence $\int_0^1 \frac{15x^3}{\sqrt{1-x^8}}dx = \frac{15\pi}{8}$.

Space for Hints

4.4. Properties of Definite integral

Definition : A function $f(x)$ is called an even function if $f(-x)=f(x)$ and a function $f(x)$ is called an odd function if $f(-x)=-f(x)$.

Property 1 :

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Proof : Let $\int f(x)\,dx = F(x)+c$

$$\therefore \int_a^b f(x)\,dx = F(b)-F(a)$$

---=

$$-\left[F(a)-F(b)\right]$$

---=

$$-\int_b^a f(x)\,dx$$

Thus $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

This proves the property (1)

Property 2 : $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a<c<b$.

Proof : Let $\int f(x)\,dx = F(x)+c$

$$\therefore \int_a^b f(x)\,dx = F(b)-F(a)$$

$$= \left[F(b)-F(c)\right]+\left[F(c)-F(a)\right]$$

$$= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Hence $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

This proves the property (2).

Property 3 : $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is an even function} \\ 0 & \text{if } f(x) \text{ is an odd function} \end{cases}$

Proof :

Case (1) : Let $f(x)$ be an even function

(i.e) $f(-x) = f(x)$ -- (4.25)

Now $\int_{-a}^{a} f(x)\,dx$

$= \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$ (using property (2))

$= I_1 + I_2$ -- (4.26)

where $I_1 = \int_{-a}^{0} f(x)\,dx$ and $I_2 = \int_0^a f(x)\,dx$.

Now $I_1 = \int_{-a}^{0} f(x)\,dx$

put $x = -y$

$\therefore\ dx = -dy$

When $x = -a$ then $y = a$

When $x = 0$ then $y = 0$

Thus $I_1 = \int_{y=a}^{0} f(-y)(-dy)$

Space for Hints

Space for Hints

$$= -\int_{y=a}^{0} f(y)\,dy \quad \text{(from (4.25))}$$

$$= \int_{0}^{a} f(y)\,dy \quad \text{(using property (1))}$$

$$= \int_{0}^{a} f(x)\,dx \quad \text{(changing the variable name as } x\text{)}$$

Thus (4.26) $\Rightarrow$ I = $I_1 + I_2$

$$= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= 2\int_{0}^{a} f(x)\,dx$$

Hence if $f(x)$ is an even function then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

Case (2) : Let $f(x)$ be an odd function

(i.e) $f(-x) = -f(x)$ --- (4.27)

Now $\int_{-a}^{a} f(x)\,dx$

$$= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \quad \text{(using property (2))}$$

$$= I_1 + I_2 \quad \text{--- (4.28)}$$

where $I_1 = \int_{-a}^{0} f(x)\,dx$ and $I_2 = \int_{0}^{a} f(x)\,dx$.

Now $I_1 = \int_{-a}^{0} f(x)\,dx$

put $x = -y$

$\therefore\ dx = -dy$

When $x=-a$ then $y=a$

When $x=0$ then $y=0$

Thus $I_1 = \int_{y=a}^{0} f(-y)(-dy)$

$$= -\int_{y=a}^{0} \left(-f(y)\right) dy \quad \text{(from (4.27))}$$

$$= -\int_{0}^{a} f(y)\, dy \quad \text{(using property (1))}$$

$$= -\int_{0}^{a} f(x)\, dx \quad \text{(changing the variable name as } x\text{)}$$

Thus (4.28) $\Rightarrow$ $I = I_1 + I_2$

$$= -\int_{0}^{a} f(x)\, dx + \int_{0}^{a} f(x)\, dx$$

$$= 0$$

Hence if $f(x)$ is an odd function then $\int_{-a}^{a} f(x)\, dx = 0$

Thus
$$\int_{-a}^{a} f(x)\, dx = \begin{cases} 2\int_{0}^{a} f(x)\, dx & \text{if } f(x) \text{ is an even function} \\ 0 & \text{if } f(x) \text{ is an odd function} \end{cases}$$

Property 4 : $\int_{0}^{a} f(x)\, dx = \int_{0}^{a} f(a-x)\, dx$

Proof : RHS $= \int_{0}^{a} f(a-x)\, dx$

put $y = a - x$

$\therefore\ dy = -dx$

When $x=0$ then $y=a$

When $x=a$ then $y=0$

Space for Hints

Space for Hints

Thus RHS = $\int_0^a f(a-x)\,dx$

$= \int_a^0 f(y)(-dy)$

$= -\int_a^0 f(y)\,dy$

$= \int_0^a f(y)\,dy$

$= \int_0^a f(x)\,dx$ (by changing the variable name as x)

Hence $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

Property 5 : $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(x) = f(2a-x)$

Proof : Let $f(x) = f(2a-x)$ -- (4.29)

$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$ (using property (2))

$= I_1 + I_2$ -- (4.30)

where $I_1 = \int_0^a f(x)\,dx$ and $I_2 = \int_a^{2a} f(x)\,dx$.

Now $I_2 = \int_a^{2a} f(x)\,dx$

put $y = 2a - x$

$\therefore\ dy = -dx$

When $x = a$ then $y = a$

When $x = 2a$ then $y = 0$

Space for Hints

Then $I_2 = \int_a^{2a} f(x)\,dx$

$$= \int_{y=a}^{0} f(2a-y)(-dy)$$

$$= -\int_{y=a}^{0} f(y)\,dy \text{ (using (4.29))}$$

$$= \int_{y=0}^{a} f(y)\,dy \text{ (using property (1))}$$

$$= \int_0^a f(x)\,dx \text{ (by changing the variable name to } x\text{)}$$

Now from (4.30), we have,

$$I = I_1 + I_2$$

$$= \int_0^a f(x)\,dx + \int_0^a f(x)\,dx$$

$$= 2\int_0^a f(x)\,dx$$

Hence $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(x) = f(2a-x)$

This proves the property (5)

Example 4.5.1 :

Evaluate $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

Solution :

$$\text{Let I} = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx \quad \text{--} \quad (4.31)$$

Space for Hints

$$= \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\,dx \quad \text{(using property (4))}$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \text{ -- (4.32)}$$

Now (4.31) + (4.32) $\Rightarrow$

$$\mathrm{I}+\mathrm{I} = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx$$

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$= \int_0^{\pi/2} dx$$

$$= \left[x\right]_0^{\pi/2}$$

$$= \frac{\pi}{2}-0$$

$$= \frac{\pi}{2}$$

Thus $I = \frac{\pi}{4}$

(i.e) $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \frac{\pi}{4}$

Example 4.5.2 :

Show that $\displaystyle\int_0^{\pi/4} \log(1+\tan x)\,dx = \frac{\pi}{8}\log 2$

Solution :

Let $\mathrm{I} = \displaystyle\int_0^{\pi/4} \log(1+\tan x)\,dx$ -- (4.33)

$$= \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$$

Now $\tan\left(\frac{\pi}{4}-x\right) = \dfrac{\tan\left(\frac{\pi}{4}\right)-\tan x}{1+\tan\left(\frac{\pi}{4}\right)\cdot\tan x}$

$$= \frac{1-\tan x}{1+\tan x}$$

and $1+\tan\left(\frac{\pi}{4}-x\right) = 1+\dfrac{1-\tan x}{1+\tan x}$

$$= \frac{1+\tan x+1-\tan x}{1+\tan x}$$

$$= \frac{2}{1+\tan x}$$

$$\log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right) = \log\left(\frac{2}{1+\tan x}\right)$$

Thus Let I $= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx$ -------------------------------------- (4.34)

Now (4.33) + (4.34) $\Rightarrow$

$$I + I = \int_0^{\pi/4} \log(1+\tan x)\,dx + \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx$$

$$2I = \int_0^{\pi/4}\left[\log(1+\tan x)+\log\left(\frac{2}{1+\tan x}\right)\right]dx$$

$$= \int_0^{\pi/4} \log 2\,dx$$

$$= \log 2 \int_0^{\pi/4} dx$$

$$= \log 2\left[x\right]_0^{\pi/4}$$

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$$= \log 2\left[\frac{\pi}{4} - 0\right]$$

$$= \frac{\pi}{4}\log 2$$

Thus $I = \frac{\pi}{8}\log 2$

(i.e) $\int_0^{\pi/4} \log(1+\tan x)\,dx = \frac{\pi}{8}\log 2$

This proves the problem.

Example 4.5.3 :

Find the value of $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$

Solution :

Let I $= \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$ --- (4.35)

Now I $= \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$

$$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x)}dx$$

$$= \int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x}dx - \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$$

$$= \int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x}dx - I \quad \text{(using (4.35))}$$

$\therefore$ 2I $= \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$ --- (4.36)

put $t = \cos x$

$\therefore\ dt = -\sin x\, dx$

When $x = 0$ then $t = \cos 0 = 1$

When $x = \pi$ then $t = \cos \pi = -1$

Thus (4.36) $\Rightarrow$ $$2\mathrm{I} = \pi \int_{1}^{-1} \frac{-dt}{1+t^2}$$

$$= \pi \int_{-1}^{1} \frac{dt}{1+t^2}$$

$$= 2\pi \int_{0}^{1} \frac{dt}{1+t^2} \text{ since the integrant is an even function}$$

$$= 2\pi \left[\tan^{-1} t \right]_0^1$$

$$= 2\pi \left[\tan^{-1}(1) - \tan^{-1}(0) \right]$$

$$= 2\pi \left[\frac{\pi}{4} - 0 \right]$$

$$= \frac{\pi^2}{2}$$

Hence $\displaystyle\int_{0}^{\pi} \frac{x \sin x}{1+\cos^2 x} dx = \frac{\pi^2}{2}$.

Example 4.5.4 :

Evaluate $\displaystyle\int_{0}^{\pi/2} \log \sin x\, dx$

Solution : Let $\mathrm{I} = \displaystyle\int_{0}^{\pi/2} \log \sin x\, dx$ -- (4.37)

$$\therefore\ \mathrm{I} = \int_{0}^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx$$

$$= \int_{0}^{\pi/2} \log \cos x\, dx \quad \text{-- (4.38)}$$

Space for Hints

Space for Hints

Now (4.37) + (4.38) $\Rightarrow$ $\mathrm{I}+\mathrm{I} = \int_0^{\pi/2} \log\sin x\,dx + \int_0^{\pi/2} \log\cos x\,dx$

$$2\mathrm{I} = \int_0^{\pi/2} (\log\sin x + \log\cos x)\,dx$$

$$= \int_0^{\pi/2} \log(\sin x\cos x)\,dx$$

$$= \int_0^{\pi/2} \log\left(\frac{2}{2}\sin x\cos x\right)dx$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right)dx$$

$$= \int_0^{\pi/2} (\log\sin 2x - \log 2)\,dx$$

$$= \int_0^{\pi/2} \log\sin 2x\,dx - \int_0^{\pi/2} \log 2\,dx$$

$$= I_1 - I_2$$

where $I_1 = \int_0^{\pi/2} \log\sin 2x\,dx$ and $I_2 = \int_0^{\pi/2} \log 2\,dx$

Now $I_1 = \int_0^{\pi/2} \log\sin 2x\,dx$

put $t = 2x$

$\therefore\ dt = 2\,dx \Rightarrow dx = \dfrac{dt}{2}$

When $x = 0$ then $t = 0$

When $x = \dfrac{\pi}{2}$ then $t = \pi$

$$\therefore\ I_1 = \int_{t=0}^{\pi} \log\sin t\left(\frac{dt}{2}\right)$$

Space for Hints

$$= \frac{1}{2} \int_{t=0}^{\pi} \log \sin t \, dt$$

$$= \frac{1}{2} 2 \int_{t=0}^{\pi/2} \log \sin t \, dt \quad (\because \log \sin(\pi - x) = \log \sin x \text{ and by property (5)})$$

$$= \int_{t=0}^{\pi/2} \log \sin t \, dt$$

$$= \text{I}$$

and $I_2 = \int_0^{\pi/2} \log 2 \, dx$

$$= \log 2 \int_0^{\pi/2} dx$$

$$= \log 2 \left[x \right]_0^{\pi/2}$$

$$= \log 2 \left[\frac{\pi}{2} - 0 \right]$$

$$= \frac{\pi}{2} \log 2$$

Now 2I = $I_1 - I_2$

(i.e) 2I = $I - \frac{\pi}{2} \log 2$

$\therefore$ I = $\frac{\pi}{2} \log \left(\frac{1}{2} \right)$

(i.e) $\int_0^{\pi/2} \log \sin x \, dx = \frac{\pi}{2} \log \left(\frac{1}{2} \right)$.

Example 4.5.5 :

Evaluate $\int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x} dx$

Solution : Let I = $\int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x} dx$ ------------------------------------ (4.39)

Space for Hints

$$\therefore\ I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right)+\sin\left(\frac{\pi}{2}-x\right)}dx$$

$$= \int_0^{\pi/2} \frac{\cos x}{\sin x+\cos x}dx \quad \text{--} \ (4.40)$$

Now (4.39) + (4.40) $\Rightarrow I + I = \int_0^{\pi/2} \frac{\sin x}{\cos x+\sin x}dx + \int_0^{\pi/2} \frac{\cos x}{\sin x+\cos x}dx$

$$\text{(i.e)}\ 2I = \int_0^{\pi/2} \frac{\sin x+\cos x}{\cos x+\sin x}dx$$

$$= \int_0^{\pi/2} dx$$

$$= \left[x\right]_0^{\pi/2}$$

$$= \left[\frac{\pi}{2}-0\right]$$

$$= \frac{\pi}{2}$$

$$\therefore\ I = \frac{\pi}{4}$$

$$\text{(i.e)} \int_0^{\pi/2} \frac{\sin x}{\cos x+\sin x}dx = \frac{\pi}{4}$$

Example 4.5.6 :

Evaluate $\int_{-1}^{1} \frac{x^2 \sin^{-1}x}{\sqrt{1-x^2}}dx$

Solution : Let $I = \int_{-1}^{1} \frac{x^2 \sin^{-1}x}{\sqrt{1-x^2}}dx$ -------------------------------------- (4.41)

Let $t = \sin^{-1} x$

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$\therefore \quad dt = \frac{1}{\sqrt{1-x^2}} dx$

and $x = \sin t \Rightarrow x^2 = \sin^2 t$

When $x = -1$ then $t = \sin^{-1}(-1) = -\frac{\pi}{2}$

When $x = 1$ then $t = \sin^{-1}(1) = \frac{\pi}{2}$

Thus (4.41) $\Rightarrow \mathrm{I} = \int_{-\pi/2}^{\pi/2} t \sin^2 t \, dt$

$= 0$ (since $t \sin^2 t$ is an odd function and property (3))

Example 4.5.7 :

Evaluate $\int_0^{\pi} \frac{x}{1+\cos^2 x} dx$

Solution : Let $\mathrm{I} = \int_0^{\pi} \frac{x}{1+\cos^2 x} dx$ -- (4.42)

Now $\mathrm{I} = \int_0^{\pi} \frac{\pi - x}{1+\cos^2(\pi - x)} dx$

$= \int_0^{\pi} \frac{\pi - x}{1+\cos^2 x} dx$

$= \int_0^{\pi} \frac{\pi}{1+\cos^2 x} dx - \int_0^{\pi} \frac{x}{1+\cos^2 x} dx$

$= \int_0^{\pi} \frac{\pi}{1+\cos^2 x} dx - \mathrm{I}$

(i.e) $2\mathrm{I} = \pi \int_0^{\pi} \frac{1}{1+\cos^2 x} dx$

(i.e) $2\mathrm{I} = 2\pi \int_0^{\pi/2} \frac{1}{1+\cos^2 x} dx$ (since the integrant is an even function)

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$$\text{Thus } I = \pi \int_0^{\pi/2} \frac{1}{1+\cos^2 x} dx$$

$$= \pi \int_0^{\pi/2} \frac{1}{\cos^2 x(1+\sec^2 x)} dx$$

$$= \pi \int_0^{\pi/2} \frac{\sec^2 x}{1+\sec^2 x} dx$$

$$= \pi \int_0^{\pi/2} \frac{\sec^2 x}{1+1+\tan^2 x} dx$$

$$= \pi \int_0^{\pi/2} \frac{\sec^2 x}{2+\tan^2 x} dx \quad \text{-----} \quad (4.43)$$

put $t = \tan x$

$\therefore \ dt = \sec^2 x \, dx$

When $x = 0$ then $t = \tan(0) = 0$

When $x = \frac{\pi}{2}$ then $t = \tan\left(\frac{\pi}{2}\right) = \infty$

$$\therefore (4.43) \Rightarrow I = \pi \int_0^{\infty} \frac{1}{2+t^2} dt$$

$$= \pi \int_0^{\infty} \frac{1}{\left(\sqrt{2}\right)^2 + t^2} dt$$

$$= \pi \frac{1}{\sqrt{2}} \left[\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) \right]_0^{\infty}$$

$$= \pi \frac{1}{\sqrt{2}} \left[\tan^{-1}(\infty) - \tan^{-1}(0) \right]$$

$$= \frac{\pi}{\sqrt{2}} \left[\frac{\pi}{2} - 0 \right]$$

$$= \frac{\pi}{\sqrt{2}} \cdot \frac{\pi}{2}$$

$$= \frac{\pi^2}{2\sqrt{2}}$$

Thus $\int_0^{\pi} \frac{x}{1+\cos^2 x}\,dx = \frac{\pi^2}{2\sqrt{2}}$.

Example 4.5.8 :

Evaluate $\int_0^{\pi} \frac{x}{1+\sin x}\,dx$

Solution : Let I = $\int_0^{\pi} \frac{x}{1+\sin x}\,dx$ -- (4.44)

Now I = $\int_0^{\pi} \frac{x}{1+\sin x}\,dx$

$$= \int_0^{\pi} \frac{x-\pi}{1+\sin(\pi - x)}\,dx$$

$$= \int_0^{\pi} \frac{\pi}{1+\sin x}\,dx - \int_0^{\pi} \frac{x}{1+\sin x}\,dx$$

$$= \pi \int_0^{\pi} \frac{1}{1+\sin x}\,dx - \mathrm{I}$$

(i.e) 2I $= \pi \int_0^{\pi} \frac{1}{1+\sin x}\,dx$

(i.e) 2I $= 2\pi \int_0^{\pi/2} \frac{1}{1+\sin x}\,dx$ (since $f(2a-x) = f(x)$)

Thus I $= \pi \int_0^{\pi/2} \frac{1}{1+\sin x}\,dx$

$$= \pi \int_0^{\pi/2} \frac{1}{1+\sin\left(\frac{\pi}{2}-x\right)}\,dx$$

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Space for Hints

$$= \pi \int_0^{\pi/2} \frac{1}{1+\cos x} dx$$

$$= \pi \int_0^{\pi/2} \frac{1}{2\cos^2(x/2)} dx$$

$$= \pi \int_0^{\pi/2} \frac{1}{2}\sec^2(x/2) dx$$

$$= \pi \left[\tan\left(\frac{x}{2}\right) \right]_0^{\pi/2}$$

$$= \pi \left[\tan\left(\frac{\pi}{4}\right) - \tan^{-1}(0) \right]$$

$$= \pi [1-0]$$

$$= \pi$$

Thus $\int_0^{\pi} \frac{x}{1+\sin x} dx = \pi$

Example 4.5.8 :

Show that $\int_0^{\pi/2} \log \tan x \, dx = 0$

Solution : Let I = $\int_0^{\pi/2} \log \tan x \, dx$ -- (4.45)

Now I = $\int_0^{\pi/2} \log \tan x \, dx$

$$= \int_0^{\pi/2} \log \tan\left(\frac{\pi}{2} - x\right) dx$$

$$= \int_0^{\pi/2} \log \cot x \, dx \quad \text{--} \quad (4.46)$$

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Now (4.45) + (4.46) $\Rightarrow$ $I + I = \int_0^{\pi/2} \log\tan x\, dx + \int_0^{\pi/2} \log\cot x\, dx$

(i.e) $2I = \int_0^{\pi/2} (\log\tan x + \log\cot x)\, dx$

$$= \int_0^{\pi/2} \log(\tan x \cdot \cot x)\, dx = \int_0^{\pi/2} \log(1)\, dx = 0 \text{ (since } \log(1) = 0\text{)}$$

Thus I = 0

(i.e) $\int_0^{\pi/2} \log\tan x\, dx = 0$

Check your progress

Questions :

(1) Evaluate $\int_0^{\pi} \log(1+\cos x)\, dx$, (2) Evaluate $\int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx$

(3) Show that $\int_0^{\pi/2} \frac{1}{1+\tan x} dx = \frac{\pi}{4}$

Summary

In this unit we have leaned that how to find Radius of curvature in polar coordinates, $p-r$ equation of curves, definite integral and their properties with examples.

Further Reading

You can also refer the following books for further reading.

(1) Calculus by Arumugam and Isaac

(2) Differential Calculus by Shanti Narayanan

Space for Hints

UNIT V

REDUCTION FORM OF INTEGRALS AND MULTIPLE INTEGRALS

Objectives :

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In this unit we are going to discuss reduction formulae for $\sin^n x$, $\cos^n x$, $\tan^n x$, $\operatorname{cosec}^n x$, $\sec^n x$, $\cot^n x$, $\sin^n x \cos^m x$, Bernoulli's formula, double and triple integrals.

After completing this unit, students may able to know

- o reduction formulae for $\sin^n x$, $\cos^n x$, $\tan^n x$, $\operatorname{cosec}^n x$, $\sec^n x$, $\cot^n x$, $\sin^n x \cos^m x$,
- o Bernoulli's formula
- o Double integrals
- o Triple integrals

5.1. Reduction formulae

Theorem 5.1.1 :

If n is positive integer, and if $I_n = \int \sin^n x\, dx$, then $I_n = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-1}$.

Proof : Given that n is a positive integer and $I_n = \int \sin^n x\, dx$

Now $I_n = \int \sin^n x\, dx$

$$= \int \sin^{n-1} x \sin x\, dx$$

$$= \int \sin^{n-1} x\, d(-\cos x)$$

$$= \sin^{n-1} x(-\cos x) - \int (-\cos x)(n-1)\sin^{n-2} x \cos x\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx$$

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(i.e) $I_n = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$

(i.e) $(1+n-1)I_n = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx$

(i.e) $nI_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2}$

$\therefore\ I_n = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}I_{n-1}$

This proves the theorem.

Cor 1 :

If $I_n = \int_0^{\pi/2} \sin^n x\,dx$ then $I_n = \frac{n-1}{n}I_{n-2}$

Proof : Given that $I_n = \int_0^{\pi/2} \sin^n x\,dx$

From the previous theorem, $I_n = -\frac{1}{n}\left[\sin^{n-1}x\cos x\right]_0^{\pi/2} + \frac{n-1}{n}I_{n-1}$

(i.e) $I_n = -\frac{1}{n}\left[0-0\right] + \frac{n-1}{n}I_{n-1}$

(i.e) $I_n = \frac{n-1}{n}I_{n-2}$

This proves the corollary (1).

Cor 2 :

If $I_n = \int_0^{\pi/2} \sin^n x\,dx$ then

$I_n = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\ldots I_1$ if n is odd integer and

$I_n = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\ldots I_0$ if n is even integer.

Now $I_1 = \int_0^{\pi/2} \sin x\,dx$

$= \left[-\cos x\right]_0^{\pi/2}$

$= -\left[\cos\left(\frac{\pi}{2}\right) - \cos(0)\right]$

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$$= -\left[0-(-1)\right]$$

$$= 1$$

and $I_0 = \int_0^{\pi/2} dx$

$$= \left[x\right]_0^{\pi/2}$$

$$= \left[\frac{\pi}{2} - 0\right]$$

$$= \frac{\pi}{2}$$

Thus $I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \mathrm{L} \ \frac{2}{3}$ if n is odd integer and

$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \mathrm{L} \ \frac{1}{2} \cdot \frac{\pi}{2}$ if n is even integer

Theorem 5.1.2 :

If n is positive integer, and if $I_n = \int \cos^n x \, dx$, then

$I_n = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-1}$.

Proof : Given that n is a positive integer and $I_n = \int \cos^n x \, dx$

Now $I_n = \int \cos^n x \, dx$

$$= \int \cos^{n-1} x \cos x \, dx$$

$$= \int \cos^{n-1} x \, d(\sin x)$$

$$= \cos^{n-1} x (\sin x) - \int (\sin x)(n-1) \cos^{n-2} x (-\sin x) \, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \sin^2 x \, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) \, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx$$

(i.e) $I_n = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx$

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(i.e) $(1+n-1)I_n = \cos^{n-1}x\sin x + (n-1)\int\cos^{n-2}x\, dx$

(i.e) $nI_n = \cos^{n-1}x\sin x + (n-1)I_{n-2}$

$\therefore\ I_n = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}I_{n-1}$

This proves the theorem.

Cor 1 :

If $I_n = \int_0^{\pi/2}\cos^n x\, dx$ then $I_n = \frac{n-1}{n}I_{n-2}$

Proof : Given that $I_n = \int_0^{\pi/2}\cos^n x\, dx$

From the previous theorem, $I_n = \frac{1}{n}\left[\cos^{n-1}x\sin x\right]_0^{\pi/2} + \frac{n-1}{n}I_{n-1}$

(i.e) $I_n = \frac{1}{n}\left[0-0\right] + \frac{n-1}{n}I_{n-1}$

(i.e) $I_n = \frac{n-1}{n}I_{n-2}$.

This proves the corollary (1).

Cor 2 :

If $I_n = \int_0^{\pi/2}\cos^n x\, dx$ then

$I_n = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\ldots I_1$ if n is odd integer and

$I_n = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\ldots I_0$ if n is even integer.

Now $I_1 = \int_0^{\pi/2}\cos x\, dx$

$= \left[\sin x\right]_0^{\pi/2}$

$= \sin\left(\frac{\pi}{2}\right) - \sin(0)$

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$$= 1 - 0$$

$$= 1$$

and $I_0 = \int_0^{\pi/2} dx$

$$= \left[x \right]_0^{\pi/2}$$

$$= \left[\frac{\pi}{2} - 0 \right]$$

$$= \frac{\pi}{2}$$

Thus $I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3}$ if n is odd integer and

$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2}$ if n is even integer

This proves the corollary.

Example 5.1.1 :

Evaluate $\int_0^{\pi/2} \cos^{10} x \, dx$.

Solution : Let $I_{10} = \int_0^{\pi/2} \cos^{10} x \, dx$

We know that $I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2}$ if n is even integer

$$\therefore \; I_{10} = \frac{10-1}{10} \cdot \frac{10-3}{10-2} \cdot \frac{10-5}{10-4} \cdot \frac{10-7}{10-6} \cdot \frac{10-9}{10-8} \cdot \frac{\pi}{2}$$

(i.e) $I_{10} = \frac{9}{10} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$

Example 5.1.2 :

Evaluate $\int_0^{\pi/2} \sin^{11} x \, dx$

Solution : Let $I_{11} = \int_0^{\pi/2} \sin^{11} x \, dx$

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We know that $I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \text{L} \ \frac{2}{3}$ if n is odd integer

$$\therefore \ I_{11} = \frac{11-1}{11} \cdot \frac{11-3}{11-2} \cdot \frac{11-5}{11-4} \cdot \frac{11-7}{11-6} \cdot \frac{11-9}{11-8}$$

(i.e) $I_{11} = \frac{10}{11} \cdot \frac{8}{9} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3}$.

Theorem 5.1.3 :

If n is positive integer, and if $I_n = \int \tan^n x \, dx$, then $I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$.

Proof : Given that $I_n = \int \tan^n x \, dx$

Now $I_n = \int \tan^n x \, dx$

$$= \int \tan^{n-2} x \tan^2 x \, dx$$

$$= \int \tan^{n-2} x (\sec^2 x - 1) \, dx$$

$$= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$$

$$= \int \tan^{n-2} x \, d(\tan x) - I_{n-2}$$

$$= \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

(i.e) $I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$.

This proves the theorem.

Cor :

The ultimate integral of $I_n = \int \tan^n x \, dx$ is given below :

When n is even then $I_0 = \int dx = x$ and

When n is odd then $I_1 = \int \tan x \, dx = \log \sec x$

Example 5.1.3 :

Evaluate $\int \tan^6 x \, dx$

Solution : Let $I_6 = \int \tan^6 x\,dx$

We know that if n is positive integer, and if $I_n = \int \tan^n x\,dx$, then

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2} \text{ and } I_0 = x.$$

$$\therefore\ I_6 = \frac{\tan^{6-1} x}{6-1} - I_{6-2}$$

(i.e) $I_6 = \dfrac{\tan^5 x}{5} - I_4$

$$= \frac{\tan^5 x}{5} - \left[\frac{\tan^3 x}{3} - I_2\right]$$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + I_2$$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - I_0$$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + c$$

Example 5.1.4 :

Evaluate $\int_0^{\pi/4} \tan^5 x\,dx$

Solution : Let $I_5 = \int_0^{\pi/4} \tan^5 x\,dx$

We know that if n is positive integer, and if $I_n = \int \tan^n x\,dx$, then

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2} \text{ and } I_1 = \log \sec x.$$

Now $I_5 = \int_0^{\pi/4} \tan^5 x\,dx$

$$\therefore\ I_5 = \left[\frac{\tan^{5-1} x}{5-1}\right]_0^{\pi/4} - I_{5-2}$$

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Space for Hints

(i.e) $I_5 = \left(\frac{1}{4} - 0\right) - I_3$

$$= \frac{1}{4} - \left[\left[\frac{\tan^{3-1} x}{3-1}\right]_0^{\pi/4} - I_{3-2}\right]$$

$$= \frac{1}{4} - \frac{1}{2} + I_1$$

$$= -\frac{1}{4} + \left[\log \sec x\right]_0^{\pi/4}$$

$$= -\frac{1}{4} + \left[\log \sec\left(\frac{\pi}{4}\right) - \log \sec(0)\right]$$

$$= -\frac{1}{4} + \left[\log \sqrt{2} - 0\right]$$

$$= \frac{1}{2}\log 2 - \frac{1}{4}$$

Theorem 5.1.4 :

If n is positive integer, and if $I_n = \int \cot^n x\, dx$, then $I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$.

Proof : Given that $I_n = \int \cot^n x\, dx$

Now $I_n = \int \cot^n x\, dx$

$$= \int \cot^{n-2} x \cot^2 x\, dx$$

$$= \int \cot^{n-2} x (\operatorname{cosec}^2 x - 1)\, dx$$

$$= \int \cot^{n-2} x \operatorname{cosec}^2 x\, dx - \int \cot^{n-2} x\, dx$$

$$= \int \cot^{n-2} x\, d(-\cot x) - I_{n-2}$$

$$= -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$$

(i.e) $I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$.

This proves the theorem.

Cor :

The ultimate integral of $I_n = \int \cot^n x\,dx$ is given below :

When n is even then $I_0 = \int dx = x$ and

When n is odd then $I_1 = \int \cot x\,dx = \log \sin x$

Example 5.1.5 :

Evaluate $\int \cot^3 x\,dx$

Solution : Let $I_3 = \int \cot^3 x\,dx$

We know that if n is positive integer, and if $I_n = \int \cot^n x\,dx$, then

$I_n = -\dfrac{\cot^{n-1} x}{n-1} - I_{n-2}$ and $I_1 = \log \sin x$.

Now $I_3 = \int \cot^3 x\,dx$

$$\therefore\ I_3 = -\frac{\cot^{3-1} x}{3-1} - I_{3-2}$$

$$\text{(i.e)}\ \ I_3 = -\frac{\cot^2 x}{2} - I_1$$

$$= -\frac{\cot^2 x}{2} - \log \sin x + c$$

Example 5.1.6 :

Evaluate $\int \cot^4 x\,dx$

Solution : Let $I_4 = \int \cot^4 x\,dx$

We know that if n is positive integer, and if $I_n = \int \cot^n x\,dx$, then

$I_n = -\dfrac{\cot^{n-1} x}{n-1} - I_{n-2}$ and $I_1 = x$.

Now $I_4 = \int \cot^4 x\,dx$

$$\therefore\ I_4 = -\frac{\cot^{4-1} x}{4-1} - I_{4-2}$$

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Space for Hints

(i.e) $I_4 = -\frac{\cot^3 x}{3} - I_2$

$$= -\frac{\cot^3 x}{3} - \left[-\frac{\cot^{2-1} x}{2-1} - I_{2-2}\right]$$

$$= -\frac{\cot^3 x}{3} + \cot x + I_0$$

$$= -\frac{\cot^3 x}{3} + \cot x + x + c$$

Theorem 5.1.5 :

If n is positive integer, and if $I_n = \int \sec^n x \, dx$, then $I_n = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1} I_{n-2}$.

Proof : Given that $I_n = \int \sec^n x \, dx$

Now $I_n = \int \sec^n x \, dx$

$$= \int \sec^{n-2} x \sec^2 x \, dx$$

$$= \int \sec^{n-2} x \, d(\tan x)$$

$$= \sec^{n-2} x \tan x - \int \tan x (n-2) \sec^{n-3} x \sec x \tan x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \tan^2 x \sec^{n-2} x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x \, dx + (n-2) \int \sec^{n-2} x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

(i.e) $I_n = \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$

(i.e) $(n-2+1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

(i.e) $(n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

(i.e) $I_n = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1} I_{n-2}$.

This proves the theorem.

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Cor :

The ultimate integral of $I_n = \int \sec^n x\, dx$ is given below :

When n is even then $I_0 = \int dx = x$ and

When n is odd then $I_1 = \int \sec x\, dx = \log(\tan x + \sec x)$

Example 5.1.7 :

Evaluate $\int_0^{\pi/4} \sec^5 x\, dx$

Solution : Let $I_5 = \int_0^{\pi/4} \sec^5 x\, dx$

We know that if n is positive integer, and if $I_n = \int \tan^n x\, dx$, then

$I_n = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1} I_{n-2}$ and $I_1 = \log(\tan x + \sec x)$.

Now $I_5 = \int_0^{\pi/4} \sec^5 x\, dx$

$$\therefore\ I_5 = \left[\frac{1}{5-1}\sec^{5-2} x \tan x\right]_0^{\pi/4} + \frac{5-2}{5-1} I_{5-2}$$

$$\text{(i.e)}\ \ I_5 = \left[\frac{1}{4}\sec^3 x \tan x\right]_0^{\pi/4} + \frac{3}{4} I_3$$

$$= \left[\frac{1}{4}\sec^3\left(\frac{\pi}{4}\right)\cdot 1 - 0\right] + \frac{3}{4}\left[\left[\frac{\sec^{3-2} x \tan x}{3-1}\right]_0^{\pi/4} + \frac{3-2}{3-1} I_{3-2}\right]$$

$$= \frac{1}{4}\left(\sqrt{2}\right)^3 + \frac{3}{4}\left[\left[\frac{\sec x \tan x}{2}\right]_0^{\pi/4} + \frac{1}{2} I_{3-2}\right]$$

$$= \frac{1}{4}\cdot 2\cdot\sqrt{2} + \frac{3}{8}\left(\sqrt{2}\right) + \frac{3}{8} I_1$$

$$= \frac{1}{2}\cdot\sqrt{2} + \frac{3}{8}\left(\sqrt{2}\right) + \frac{3}{8}\left[\log(\tan x + \sec x)\right]_0^{\pi/4}$$

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$$= \frac{7\sqrt{2}}{8} + \frac{3}{8}\left[\log\left(1+\sqrt{2}\right) - 0\right]$$

$$= \frac{7\sqrt{2}}{8} + \frac{3}{8}\log\left(1+\sqrt{2}\right)$$

Theorem 5.1.6 :

If n is positive integer, and if $I_n = \int \operatorname{cosec}^n x\, dx$, then $I_n = -\frac{1}{n-1}\operatorname{cosec}^{n-2} x \cot x + \frac{n-2}{n-1} I_{n-2}$.

Proof : Given that $I_n = \int \operatorname{cosec}^n x\, dx$

Now $I_n = \int \operatorname{cosec}^n x\, dx$

$$= \int \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x\, dx$$

$$= \int \operatorname{cosec}^{n-2} x\; d(-\cot x)$$

$$= -\operatorname{cosec}^{n-2} x \cot x - \int \cot x\,(n-2)\operatorname{cosec}^{n-3} x \operatorname{cosec} x \cot x\, dx$$

$$= -\operatorname{cosec}^{n-2} x \cot x - (n-2)\int \operatorname{cosec}^{n-2} x \cot^2 x\, dx$$

$$= -\operatorname{cosec}^{n-2} x \cot x - (n-2)\int \operatorname{cosec}^{n-2} x(\operatorname{cosec}^2 x - 1)\, dx$$

$$= -\operatorname{cosec}^{n-2} x \cot x - (n-2)\int \operatorname{cosec}^{n} x\, dx + (n-2)\int \operatorname{cosec}^{n-2} x\, dx$$

$$= -\operatorname{cosec}^{n-2} x \cot x - (n-2) I_n + (n-2) I_{n-2}$$

(i.e) $I_n = -\operatorname{cosec}^{n-2} x \cot x - (n-2) I_n + (n-2) I_{n-2}$

(i.e) $(n-2+1) I_n = -\operatorname{cosec}^{n-2} x \cot x + (n-2) I_{n-2}$

(i.e) $(n-1) I_n = -\operatorname{cosec}^{n-2} x \cot x + (n-2) I_{n-2}$

(i.e) $I_n = -\frac{1}{n-1}\operatorname{cosec}^{n-2} x \cot x + \frac{n-2}{n-1} I_{n-2}$

This proves the theorem.

Cor :

The ultimate integral of $I_n = \int \sec^n x\, dx$ is given below :

When n is even then $I_0 = \int dx = x$ and

When n is odd then $I_0 = \int \operatorname{cosec} x \, dx = -\log(\operatorname{cosec} x + \cot x)$

Theorem 5.1.6 :

If m, n are positive integers, and if $I_{m,n} = \int \sin^m x \cos^n x \, dx$, then

$$I_{m,n} = \frac{1}{m+n} \cos^{n-1} x \sin^{m+1} x + \frac{n-1}{m+n} I_{m,n-2}.$$

Proof : Given that m, n are positive integers, and if $I_{m,n} = \int \sin^m x \cos^n x \, dx$.

$$\text{Now } I_{m,n} = \int \sin^m x \cos^n x \, dx$$

$$= \int \sin^m x \cos^{n-1} x \cos x \, dx$$

$$= \int \sin^m x \cos^{n-1} x \, d(\sin x)$$

$$= \int \cos^{n-1} x \, d\left(\frac{\sin^{m+1} x}{m+1}\right)$$

$$= \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} - \int \frac{1}{m+1} \sin^{m+1} x (n-1) \cos^{n-2} x (-\sin x) \, dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{m+2} x \cos^{n-2} x \, dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^m x \cos^{n-2} x \sin^2 x \, dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \sin^m x \cos^{n-2} x (1 - \cos^2 x) \, dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n}$$

$$\text{(i.e) } I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n}$$

$$\therefore \left(1 + \frac{n-1}{m+1}\right) I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$\text{(i.e) } \left(\frac{m+n}{m+1}\right) I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

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(i.e) $(m+n)I_{m,n} = \cos^{n-1}x\sin^{m+1}x+(n-1)I_{m,n-2}$

Thus $I_{m,n} = \dfrac{1}{m+n}\cos^{n-1}x\sin^{m+1}x+\dfrac{n-1}{m+n}I_{m,n-2}$ ----------------- (0.1)

This proves the theorem.

Note (1) :

In the above theorem we reduced the power of $\cos x$ by 2. Similarly we may reduce the power of $\sin x$ by 2 and we get

$I_{m,n} = -\dfrac{1}{m+n}\sin^{m-1}x\cos^{n+-1}x+\dfrac{m-1}{m+n}I_{\mathrm{m}-2,n}$ ------------------- (0.2)

Note (2) :

Let m or n are odd integers, say n, then using (0.1), we have the ultimate integral as

$$I_{m,1} = \int \sin^m x\cos x\,dx$$

$$= \int \sin^m x\,d(\sin x)$$

$$= \frac{\sin^{m+1}x}{m+1}$$

Note (3) : Let m or n are odd integers, say m, then using (0.2), we have the ultimate integral as

$$I_{1,n} = \int \sin x\cos^n x\,dx$$

$$= \int \cos^n x\,d(-\cos x)$$

$$= -\frac{\cos^{n+1}x}{n+1}$$

Note (4) :

Let m and n are even integers and $m<n$, then using (0.2), we have the ultimate integral as

$I_{0,n} = \int \cos^n x\,dx$.

We know the method of integrating the reduction formula of $\int \cos^n x\,dx$.

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Corollary : Let m and n are positive integers, then we shall find

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx.$$

Now from (0.1), we have,

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx$$

$$= \frac{1}{m+n}\left[\cos^{n-1} x \sin^{m+1} x\right]_0^{\pi/2} + \frac{n-1}{m+n} I_{m,n-2}$$

$$= \frac{1}{m+n}\left[0-0\right] + \frac{n-1}{m+n} I_{m,n-2}$$

$$= \frac{n-1}{m+n} I_{m,n-2}$$

$$= \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} I_{m,n-4}$$

$$\vdots \quad \vdots \quad \vdots$$

$$= \begin{cases} \dfrac{n-1}{m+n} \cdot \dfrac{n-3}{m+n-2} \cdot \dfrac{n-5}{m+n-4} \cdots I_{m,1} & \text{if } n \text{ is odd} \\[2ex] \dfrac{n-1}{m+n} \cdot \dfrac{n-3}{m+n-2} \cdot \dfrac{n-5}{m+n-4} \cdots I_{m,0} & \text{if } n \text{ is even} \end{cases} \qquad \text{----- (0.3)}$$

Case (i) :

If n is odd, then $I_{m,1} = \displaystyle\int_0^{\pi/2} \sin^m x \cos x \, dx$

$$= \left[\frac{\sin^{m+1} x}{m+1}\right]_0^{\pi/2}$$

$$= \frac{1}{m+1}$$

Thus $\displaystyle\int_0^{\pi/2} \sin^m x \cos^n x \, dx$

$$= \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdot \frac{n-5}{m+n-4} \cdot \cdots \frac{1}{m+1} \text{ if } m \text{ is even and } n \text{ is any integer.}$$

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Case (ii) :

If n is even, then $I_{m,0} = \int_0^{\pi/2} \sin^m x\,dx$

$$= \frac{m-1}{m}\cdot\frac{m-3}{m-2}\cdot\frac{m-5}{m-4}\cdot\mathrm{L}\ \cdot\frac{1}{2}\cdot\frac{\pi}{2} \quad \text{if } m \text{ is even}$$

Thus $\int_0^{\pi/2} \sin^m x \cos^n x\,dx$

$$= \frac{n-1}{m+n}\cdot\frac{n-3}{m+n-2}\cdot\frac{n-5}{m+n-4}\cdot\mathrm{L}\ \frac{1}{m+1}\cdot\frac{m-1}{m}\cdot\frac{m-3}{m-2}\cdot\frac{m-5}{m-4}\cdot\mathrm{L}\ \cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

if m is even and n is any integer.

Example 5.1.8 :

Evaluate $\int_0^{\pi/2} \sin^7 x \cos^5 x\,dx$

Solution : Let I = $\int_0^{\pi/2} \sin^7 x \cos^5 x\,dx$

(i.e) $I = \frac{6\cdot4\cdot2\cdot4\cdot2}{12\cdot10\cdot8\cdot6\cdot4\cdot2}$

(i.e) $I = \frac{1}{120}$

Example 5.1.9 :

Evaluate $\int_0^{\pi/2} \sin^6 x \cos^8 x\,dx$

Solution : Let I = $\int_0^{\pi/2} \sin^6 x \cos^8 x\,dx$

(i.e) $I = \frac{5\cdot3\cdot1\cdot7\cdot5\cdot3\cdot1}{14\cdot12\cdot10\cdot8\cdot6\cdot4\cdot2}\cdot\frac{\pi}{2}$

(i.e) $I = \frac{5\pi}{2}$

Example 5.1.10 :

Evaluate $\int_0^{\pi/2} \sin^6 x \cos^5 x \, dx$

Solution : Let $I = \int_0^{\pi/2} \sin^6 x \cos^5 x \, dx$

(i.e) $I = \dfrac{4\cdot 2\cdot 5\cdot 3\cdot 1}{11\cdot 9\cdot 7\cdot 5\cdot 3}$

(i.e) $I = \dfrac{8}{693}$

Example 5.1.11 :

Evaluate $\int_0^{\pi/2} \sin^5 x \cos^6 x \, dx$

Solution : Let $I = \int_0^{\pi/2} \sin^5 x \cos^6 x \, dx$

(i.e) $I = \dfrac{5\cdot 3\cdot 1\cdot 4\cdot 2}{11\cdot 9\cdot 7\cdot 5\cdot 3}$

(i.e) $I = \dfrac{8}{693}$

Example 5.1.12 :

If $f(m,n) = \int_0^{\pi/2} \cos^m x \sin nx \, dx$ prove that

$f(m,n) = \dfrac{1}{m+n} + \dfrac{m}{m+n} f(m-1, n-1)$ and deduce that

$$f(m,m) = \frac{1}{2^{m+1}}\left\{\frac{2}{1} + \frac{2^2}{2} + \frac{2^3}{3} + \mathrm{L} + \frac{2^m}{m}\right\}.$$

Proof : Given that $f(m,n) = \int_0^{\pi/2} \cos^m x \sin nx \, dx$

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$$= \int_0^{\pi/2} \cos^m x \, d\left(-\frac{\cos nx}{n}\right)$$

$$= -\frac{1}{n}\int_0^{\pi/2} \cos^m x \, d(\cos nx)$$

$$= -\frac{1}{n}\left[\cos^m x \cdot \cos nx\right]_0^{\pi/2} + \frac{1}{n}\int_a^b \cos nx\,(m)\cos^{m-1} x(-\sin x)\,dx$$

$$= \frac{1}{n} - \frac{m}{n}\int_0^{\pi/2} \cos^{m-1} x\left[\sin nx \cos x - \sin(nx - x)\right]dx$$

$$= \frac{1}{n} - \frac{m}{n}\int_0^{\pi/2} \cos^m x \sin nx \, dx + \frac{m}{n}\int_0^{\pi/2} \cos^{m-1} x \sin[(n-1)x]\,dx$$

$$= \frac{1}{n} - \frac{m}{n} f(m,n) + \frac{m}{n} f(m-1, n-1)$$

(i.e) $f(m,n) = \frac{1}{n} - \frac{m}{n} f(m,n) + \frac{m}{n} f(m-1,n-1)$

(i.e) $\left(1 + \frac{m}{n}\right) f(m,n) = \frac{1}{n} + \frac{m}{n} f(m-1,n-1)$

(i.e) $\left(\frac{m+n}{n}\right) f(m,n) = \frac{1}{n} + \frac{m}{n} f(m-1,n-1)$

Hence $f(m,n) = \frac{1}{m+n} + \frac{m}{m+n} f(m-1,n-1)$

This prove the problem.

Deduction :

Now $f(m,m) = \frac{1}{m+m} + \frac{m}{m+m} f(m-1,m-1)$

$$= \frac{1}{2m} + \frac{m}{2m} f(m-1,m-1)$$

$$= \frac{1}{2m} + \frac{1}{2}\left[\frac{1}{2(m-1)} + \frac{1}{2} f(m-2,m-2)\right]$$

$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^2} f(m-2,m-2)$$

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$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^2}\left[\frac{1}{2(m-2)} + \frac{1}{2} f(m-3, m-3)\right]$$

$$= \Lambda \quad \Lambda \quad \Lambda \quad \Lambda \quad \Lambda \quad \Lambda$$

$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^3(m-2)} + \mathrm{L} + \frac{1}{2^{m-1}} f(1,1) \text{ ------------ } (0.4)$$

Now $f(1,1) = \int_0^{\pi/2} \cos x \sin x \, dx$

$$= \frac{1}{2}\int_0^{\pi/2} \sin 2x \, dx$$

$$= \frac{1}{2}\left[-\frac{\cos 2x}{2}\right]_0^{\pi/2}$$

$$= -\frac{1}{4}\left[\cos \pi - \cos 0\right]$$

$$= -\frac{1}{4}\left[-1-1\right]$$

$$= \frac{1}{2}$$

Hence (0.4) $f(m,m) = \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^3(m-2)} + \mathrm{L} + \frac{1}{2^{m-1}}\frac{1}{2}$

$$= \frac{1}{2^{m+1}}\left[\frac{2^m}{m} + \frac{2^{m-1}}{m-1} + \frac{2^{m-2}}{m-2} + \mathrm{L} + \frac{2}{1}\right]$$

$$= \frac{1}{2^{m+1}}\left\{\frac{2}{1} + \frac{2^2}{2} + \frac{2^3}{3} + \mathrm{L} + \frac{2^m}{m}\right\}.$$

This proves the deduction.

5.2. Bernoulli's formula

We know that the formula for integration by parts as $\int u \, dv = uv - \int v \, du$. Now Bernoulli's formula is an extension of the formula of integration by parts.

Space for Hints

We denote *dashes* as successive differentiation and *suffixes* as successive integration.

(i.e) $u' = \dfrac{du}{dx}$, $u'' = \dfrac{d^2u}{dx^2}$, etc., and $v_1 = \int v\,dx$, $v_2 = \iint v(dx)^2$, etc.,

Now $\int u\,dv = uv - \int v\,du$

$$= uv - \int u'\,d(v_1)$$

$$= uv - \left[u'v_1 - \int v_1\,d(u')\right]$$

$$= uv - u'v_1 + \int v_1\,d(u')$$

$$= uv - u'v_1 + \int u''\,d(v_2)$$

$$= uv - u'v_1 + u''v_2 - \int v_2\,d(u'')$$

$$= uv - u'v_1 + u''v_2 - u'''v_3 + \text{L}$$

Example 5.2.1 :

Using Bernoulli's formula, integrate x^3e^{-2x}.

Solution : Let I = $\int x^3e^{-2x}\,dx$

(i.e) I = $\int u\,dv$ where $u = x^3$ and $v = e^{-2x}$

Now $u = x^3$

$\therefore\ u' = 3x^2$, $u'' = 6x$, $u''' = 6$ and $u'''' = 0$

and $v = e^{-2x}$,

$$v_1 = \int e^{-2x}\,dx$$

$$= \frac{e^{-2x}}{-2}$$

$$= -\frac{1}{2}e^{-2x}$$

$$v_2 = -\frac{1}{2}\int e^{-2x}\,dx$$

$$= -\frac{1}{2}\left[\frac{e^{-2x}}{-2}\right]$$

$$=\frac{1}{4}e^{-2x}$$

$$v_3=\frac{1}{4}\int e^{-2x}\,dx$$

$$=\frac{1}{4}\left[\frac{e^{-2x}}{-2}\right]$$

$$=-\frac{1}{8}e^{-2x}$$ and so on.

Thus $I = uv - u'v_1 + u''v_2 - u'''v_3 + L$

$$=x^3e^{-2x}-3x^2\left[-\frac{1}{2}e^{-2x}\right]+6x\left[\frac{1}{4}e^{-2x}\right]-6\left[-\frac{1}{8}e^{-2x}\right]+0$$

$$=e^{-2x}\left[x^3+3x^2+\frac{3}{2}x+\frac{3}{4}\right]$$

$$=\frac{1}{4}e^{-2x}\left[4^3+6x^2+6x+3\right]$$

Thus $I=\frac{1}{4}e^{-2x}\left[4^3+6x^2+6x+3\right]$.

Example 5.2.2 :

Evaluate $\int x^4 \sin x\,dx$.

Solution : Let $I = \int x^4 \sin x\,dx$

(i.e) $I = \int x^4 d(-\cos x)$

(i.e) $I = \int u\,dv$ where $u = x^4$ and $v = -\cos x$

We know that $I = uv - u'v_1 + u''v_2 - u'''v_3 + L$

(i.e) $I = x^4(-\cos x) - 4x^3(\sin x) + 12x^2(\cos x) - 24x(-\sin x) + 24(\cos x) - 0$

(i.e) $I = -x^4\cos x - 4x^3\sin x + 12x^2\cos x + 24x\sin x + 24\cos x$

Check your progress

Questions :

(1) Evaluate $\int x^4 e^x\,dx$ and (2) Evaluate $\int x^3 \sin 3x\,dx$

Space for Hints

Space for Hints

5.3. Double integral

If $f(x,y)$ is continuous on a bounded region

S = $\{(x,y)/ a \leq x \leq b$ and $\varphi_1(x) \leq y \leq \varphi_2(x)\}$ where φ_1 and φ_2 are two continuous functions defined on $[a,b]$ then the double integral is defined as

$$\iint_S f(x,y)\,dx\,dy = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy \right] dx .$$

Similarly if $f(x,y)$ is continuous on a bounded region

S = $\{(x,y)/ c \leq y \leq d$ and $\varphi_1(y) \leq x \leq \varphi_2(y)\}$ where φ_1 and φ_2 are two continuous functions defined on $[c,d]$ then the double integral is defined as

$$\iint_S f(x,y)\,dx\,dy = \int_c^d \left[\int_{\varphi_1(y)}^{\varphi_2(y)} f(x,y)\,dx \right] dy .$$

Example 5.3.1 :

Evaluate $\int_0^1 \int_0^2 x\,y^2\,dy\,dx$

Solution : Let I = $\int_0^1 \int_0^2 x\,y^2\,dy\,dx$

$$= \int_0^1 \int_0^2 x\,y^2\,dy\,dx$$

$$= \int_0^1 x \left[\frac{y^3}{3} \right]_0^2 dx$$

$$= \frac{1}{3} \int_0^1 x \left[8-0 \right] dx$$

$$= \frac{8}{3}\int_0^1 x\,dx$$

$$= \frac{8}{3}\left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{4}{3}\left[1-0\right]$$

$$= \frac{4}{3}$$

(i.e) $$\int_0^1\int_0^2 x\,y^2\,dy\,dx = \frac{4}{3}$$

Example 5.3.2 :

Evaluate $\int_0^1\int_0^2 (x^2+y^2)\,dy\,dx$

Solution : Let I = $\int_0^1\int_0^2 (x^2+y^2)\,dy\,dx$

$$= \int_0^1 \left(x^2 y + \frac{y^3}{3}\right)_0^2 dx$$

$$= \int_0^1 \left(2x^2 + \frac{2^3}{3}\right)_0^2 dx$$

$$= \left[2\frac{x^3}{3} + \frac{8}{3}x\right]_0^1$$

$$= \left[2\frac{1}{3} + \frac{8}{3}\right] - 0$$

$$= \frac{10}{3}$$

(i.e) $$\int_0^1\int_0^2 (x^2+y^2)\,dy\,dx = \frac{10}{3}$$

Space for Hints

Space for Hints

Example 5.3.3 :

Evaluate $\int_0^a \int_0^b \frac{xy}{\sqrt{1-x^2-y^2}}\, dx\, dy$.

Solution : Let I = $\int_0^a \int_0^b \frac{xy}{\sqrt{1-x^2-y^2}}\, dx\, dy$

Put $t = 1-x^2-y^2$

$\therefore\ dt = -2x\, dx$

(i.e) $x\, dx = -\frac{dt}{2}$

When $x = 0$ then $t = 1-y^2$

When $x = b$ then $t = 1-y^2-b^2$

$$\text{Thus I } = \int_0^a \int_{1-y^2}^{1-y^2-a^2} \frac{y}{\sqrt{t}}\left(-\frac{dt}{2}\right) dy$$

$$= -\frac{1}{2}\int_0^a y \int_{1-y^2}^{1-y^2-a^2} t^{-\frac{1}{2}}\, dt\, dy$$

$$= -\frac{1}{2}\int_0^a y\left[2\sqrt{t}\,\right]_{1-y^2}^{1-y^2-b^2} dy$$

$$= -\frac{2}{2}\int_0^a y\left[\sqrt{1-y^2-b^2} - \sqrt{1-y^2}\,\right] dy$$

$$= \int_0^a y\left[\sqrt{1-y^2} - \sqrt{1-y^2-b^2}\,\right] dy$$

$$= I_1 - I_2 \qquad (0.5)$$

where $I_1 = \int_0^a y\sqrt{1-y^2}\, dy$ and $I_2 = \int_0^a y\sqrt{1-y^2-b^2}\, dy$

Now $I_1 = \int_0^a y\sqrt{1-y^2}\, dy$

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Put $t = 1 - y^2$

$\therefore \; dt = -2y\,dy$

(i.e) $y\,dy = -\dfrac{dt}{2}$

When $y = 0$ then $t = 1$

When $y = a$ then $t = 1 - a^2$

$$\therefore \; I_1 = \int_1^{1-a^2} \sqrt{t}\left(-\frac{dt}{2}\right)$$

$$= -\frac{1}{2}\left[\frac{2}{3}t^{3/2}\right]_1^{1-a^2}$$

$$= -\frac{1}{2}\cdot\frac{2}{3}\cdot\left[\left(1-a^2\right)^{3/2} - 1\right]$$

$$= \frac{1}{3}\cdot\left[1 - \left(1-a^2\right)^{3/2}\right] \text{ --- (0.6)}$$

Again $I_2 = \displaystyle\int_0^a y\sqrt{1 - y^2 - b^2}\, dy$

Put $t = 1 - y^2 - b^2$

$\therefore \; dt = -2y\,dy$

(i.e) $y\,dy = -\dfrac{dt}{2}$

When $y = 0$ then $t = 1 - b^2$

When $y = a$ then $t = 1 - a^2 - b^2$

$$\therefore \; I_2 = \int_{1-b^2}^{1-a^2-b^2} \sqrt{t}\left(-\frac{dt}{2}\right)$$

$$= -\frac{1}{2}\left[\frac{2}{3}t^{3/2}\right]_{1-b^2}^{1-a^2-b^2}$$

$$= -\frac{1}{2}\cdot\frac{2}{3}\cdot\left[\left(1-a^2-b^2\right)^{3/2} - \left(1-b^2\right)^{3/2}\right]$$

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$$= \frac{1}{3}\cdot\left[\left(1-b^2\right)^{3/2} - \left(1-a^2-b^2\right)^{3/2}\right] \text{ ------------------------ } (0.7)$$

Hence, from (0.5), (0.6) and (0.7), we have,

$$I = \frac{1}{3}\cdot\left[1-\left(1-a^2\right)^{3/2}\right] - \frac{1}{3}\cdot\left[\left(1-b^2\right)^{3/2} - \left(1-a^2-b^2\right)^{3/2}\right]$$

$$\text{(i.e) } I = \frac{1}{3}\cdot\left[1-\left(1-a^2\right)^{3/2} - \left(1-b^2\right)^{3/2} + \left(1-a^2-b^2\right)^{3/2}\right]$$

Example 5.3.4 :

Evaluate $\int\limits_0^a \int\limits_{x^2/a}^{x} \frac{x}{x^2+y^2}\, dx\, dy$

Solution : Let $I = \int\limits_0^a \int\limits_{x^2/a}^{x} \frac{x}{x^2+y^2}\, dx\, dy$

$$\text{(i.e) } I = \int\limits_0^a x\left[\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right]_{x^2/a}^{x} dx$$

$$= \int\limits_0^a \left[\tan^{-1}\left(\frac{x}{x}\right) - \tan^{-1}\left(\frac{x^2/a}{x}\right)\right] dx$$

$$= \int\limits_0^a \left[\frac{\pi}{4} - \tan^{-1}\left(\frac{x}{a}\right)\right] dx$$

$$= \left[\frac{\pi}{4}x - \left\{x\tan^{-1}\left(\frac{x}{a}\right) - \frac{a}{2}\log\left(x^2+a^2\right)\right\}\right]_0^a$$

$$= \left[\frac{\pi}{4}x - x\tan^{-1}\left(\frac{x}{a}\right) + \frac{a}{2}\log\left(x^2+a^2\right)\right]_0^a$$

$$= \left[\frac{\pi}{4}a - a\tan^{-1}\left(\frac{a}{a}\right) + \frac{a}{2}\log\left(a^2+a^2\right)\right] - \left[\frac{a}{2}\log\left(a^2\right)\right]$$

$$= \left[\frac{\pi}{4}a - a\frac{\pi}{4} + \frac{a}{2}\log\left(2a^2\right)\right] - \left[\frac{a}{2}\log\left(a^2\right)\right]$$

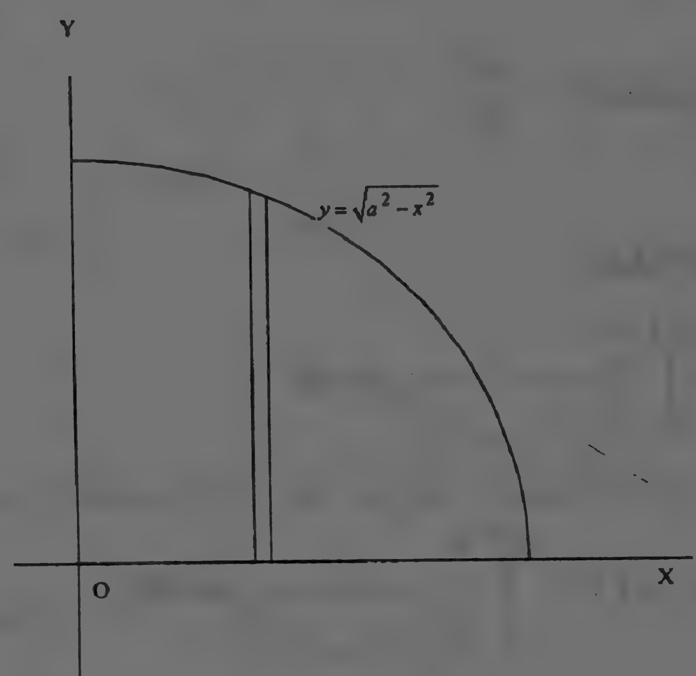
$$= \frac{a}{2}\log\left(2a^2\right) - \frac{a}{2}\log\left(a^2\right)$$

$$= \frac{a}{2}\log(2)$$

Thus $\displaystyle\int_0^a \int_{x^2/a}^{x} \frac{x}{x^2+y^2}\,dx\,dy = \frac{a}{2}\log(2)$.

Example 5.3.5 : Evaluate $\iint xy\,dx\,dy$ taken over the positive quadrant of the circle $x^2+y^2=a^2$.

Solution :



In the positive quadrant of the circle $x^2+y^2=a^2$, we have the limits as y varies from 0 to $\sqrt{a^2-x^2}$ and x varies from 0 to a.

Thus I $= \iint xy\,dx\,dy$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx.$$

$$= \int_0^a x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

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$$= \frac{1}{2}\int_0^a x\left[(a^2 - x^2) - 0\right]dx$$

$$= \frac{1}{2}\int_0^a (a^2 x - x^3)\,dx$$

$$= \frac{1}{2}\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\left(\frac{a^4}{2} - \frac{a^4}{4}\right) - 0\right]$$

$$= \frac{a^4}{8}$$

(i.e) $\iint xy\,dx\,dy = \dfrac{a^4}{8}$

Example 5.3.6 :

Evaluate $\displaystyle\int_0^{\pi/2}\int_0^{\infty}\frac{r}{\left(r^2 + a^2\right)^2}\,dr\,d\theta$

Solution : Let I = $\displaystyle\int_0^{\pi/2}\int_0^{\infty}\frac{r}{\left(r^2 + a^2\right)^2}\,dr\,d\theta$

Put $t = r^2 + a^2$

$\therefore\quad dt = 2r\,dr$

(i.e) $r\,dr = \dfrac{dt}{2}$

When $r = 0$ then $t = a^2$

When $r = \infty$ then $t = \infty$

Thus I = $\displaystyle\int_0^{\pi/2}\int_0^{\infty}\frac{1}{t^2}\frac{dt}{2}\,d\theta$

$$= \frac{1}{2}\int_{0}^{\pi/2}\left[\frac{t^{-1}}{-1}\right]_{0}^{\infty} d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi/2}\left[-\frac{1}{\infty}+\frac{1}{a^{2}}\right] d\theta$$

$$= \frac{1}{2a^{2}}\int_{0}^{\pi/2} d\theta$$

$$= \frac{1}{2a^{2}}\left[\theta\right]_{0}^{\pi/2}$$

$$= \frac{1}{2a^{2}}\left[\frac{\pi}{2}-0\right]$$

$$= \frac{\pi}{4a^{2}}$$

Hence $\displaystyle\int_{0}^{\pi/2}\int_{0}^{\infty}\frac{r}{\left(r^{2}+a^{2}\right)^{2}}\,dr\,d\theta = \frac{\pi}{4a^{2}}$.

Example 5.3.7 :

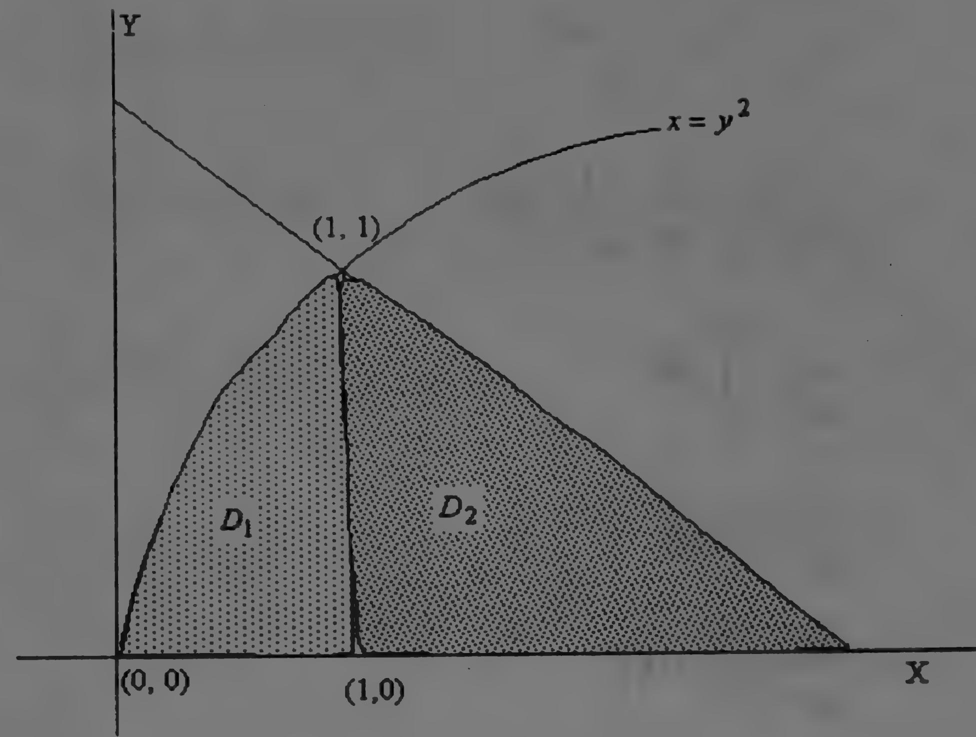
Evaluate $\displaystyle\iint_{D} xy\,dx\,dy$ where D is the region bounded by the curve $x = y^{2}$, $x = 2 - y$, $y = 0$ and $y = 1$.

Solution : Let I = $\displaystyle\iint_{D} xy\,dx\,dy$

The given region is shown in the following figure.

Space for Hints

Space for Hints



The given region D is divided into two regions D_1 and D_2.

In the regions D_1, x varies from 0 to 1 and y varies from 0 to $\sqrt{x}$ and

In the region D_2, x varies from 1 to 2 and y varies from 0 to $2-x$.

Thus I = $\iint\limits_D xy\,dx\,dy$

$$= \iint\limits_{D_1} xy\,dx\,dy + \iint\limits_{D_2} xy\,dx\,dy$$

$$= \int_0^1 \int_0^{\sqrt{x}} xy\,dx\,dy + \int_1^2 \int_0^{2-x} xy\,dx\,dy$$

$$= \int_0^1 x\left[\frac{y^2}{2}\right]_0^{\sqrt{x}} dx + \int_1^2 x\left[\frac{y^2}{2}\right]_0^{2-x} dx$$

$$= \frac{1}{2}\int_0^1 x\left[x-0\right]dx + \frac{1}{2}\int_1^2 x\left[(2-x)^2-0\right]dx$$

$$= \frac{1}{2}\int_0^1 x^2\,dx + \frac{1}{2}\int_1^2 x(4-4x+x^2)\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^1 + \frac{1}{2}\int_1^2 (4x-4x^2+x^3)\,dx$$

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$$= \frac{1}{6}\left[1-0\right] + \frac{1}{2}\left[4\frac{x^2}{2} - 4\frac{x^3}{3} + \frac{x^4}{4}\right]_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[\left(8 - \frac{32}{3} + 2\right) - \left(2 - \frac{4}{3} + \frac{1}{4}\right)\right]$$

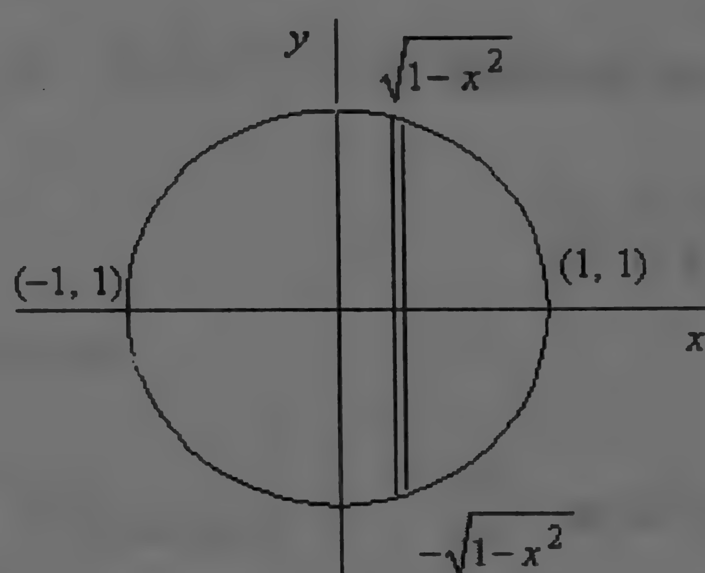
$$= \frac{9}{24}$$

Thus $I = \iint\limits_D xy\, dx\, dy = \frac{9}{24}$.

Example 5.3.8 :

Evaluate $\iint\limits_D x^2 y^2\, dx\, dy$ where D is the circular disc $x^2 + y^2 \le 1$.

Solution : Let $I = \iint\limits_D x^2 y^2\, dx\, dy$



From the figure it is clear that x varies from -1 to 1 and y varies from $-\sqrt{1-x^2}$ to $\sqrt{1-x^2}$.

Thus $I = \iint\limits_D x^2 y^2\, dx\, dy$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2 y^2\, dx\, dy$$

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$$= 4\int_0^1 \int_0^{\sqrt{1-x^2}} x^2 y^2 \, dx\, dy$$

$$= 4\int_0^1 x^2 \left[\frac{y^3}{3}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{4}{3}\int_0^1 x^2 \left[1-x^2\right]^{3/2} dx$$

Put $x = \cos\theta$

$\therefore \quad dx = -\sin\theta\, d\theta$

When $x = 0$ then $\theta = \frac{\pi}{2}$

When $x = 1$ then $\theta = 0$

$$\therefore \ \mathrm{I} = \frac{4}{3}\int_0^{\pi/2} \cos^2\theta\left[1-\cos^2\theta\right]^{3/2} (-\sin\theta)\, d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2} \cos^2\theta \sin^4\theta\, d\theta$$

$$= \frac{4}{3}\left[\frac{1}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$= \frac{\pi}{24}$$

Hence $\mathrm{I} = \iint_D x^2 y^2 \, dx\, dy = \frac{\pi}{24}$

Check your progress

Questions :

Evaluate the following

(1) $\int_0^3 \int_0^2 xy(x+y)\, dx\, dy$

(2) $\int_0^a \int_{y-a}^{2y} xy\, dx\, dy$

(3) $\int_{-\pi/2}^{\pi/2} \int_{y-a}^{2\cos\theta} r^2 \, dr \, d\theta$

(4) Evaluate $\iint_D (x-y)\, dx\, dy$ where D is the region bounded by the line $y = x$ and the parabola $x^2 = y$

(4) Evaluate $\iint_D (x+y+a)\, dx\, dy$ where D is the region bounded by the circle $x^2 + y^2 = a^2$

5.4. Triple integral

The idea of double integral can be extended to triple integral also. It can be defined as $\iiint f(x,y,z)\, dx\, dy\, dz$ as

$$\iiint f(x,y,z)\, dx\, dy\, dz = \int \left\{ \int \left[\int f(x,y,z)\, dx \right] dy \right\} dz .$$

Example 5.4.1 :

Evaluate $\int_0^a \int_0^b \int_0^c xy \, dz\, dy\, dx$

Solution : Let I = $\int_0^a \int_0^b \int_0^c xy \, dz\, dy\, dx$

$$= \left(\int_0^a x\, dx \right) \cdot \left(\int_0^b y\, dy \right) \cdot \left(\int_0^c dz \right)$$

$$= \left[\frac{x^2}{2} \right]_0^a \cdot \left[\frac{y^2}{2} \right]_0^b \cdot \left[z \right]_0^c$$

$$= \left[\frac{a^2}{2} - 0 \right] \cdot \left[\frac{b^2}{2} - 0 \right] \cdot \left[c - 0 \right]$$

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Space for Hints

$$= \frac{a^2b^2c}{4}$$

Example 5.4.1 :

Evaluate $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\,dz\,dy\,dx$

Solution : Let I = $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\,dz\,dy\,dx$

$$\text{(i.e) } I = \int_0^1 \int_{y^2}^1 x\left[z\right]_0^{1-x} dx\,dy$$

$$= \int_0^1 \int_{y^2}^1 x\left[(1-x)-0\right] dx\,dy$$

$$= \int_0^1 \int_{y^2}^1 (x-x^2)\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{y^2}^1 dy$$

$$= \int_0^1 \left[\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{y^4}{2}-\frac{y^6}{3}\right)\right] dy$$

$$= \int_0^1 \left[\frac{1}{6}-\frac{y^4}{2}+\frac{y^6}{3}\right] dy$$

$$= \left[\frac{1}{6}y-\frac{1}{2}\cdot\frac{y^5}{5}+\frac{1}{3}\cdot\frac{y^7}{7}\right]_0^1$$

$$= \left[\frac{1}{6}-\frac{1}{10}+\frac{1}{21}\right]-0$$

$$= \frac{4}{35}$$

Hence $\int\limits_0^1 \int\limits_{y^2}^1 \int\limits_0^{1-x} x\,dz\,dy\,dx = \frac{4}{35}$

Space for Hints

Example 5.4.2 :

Evaluate $\iiint\limits_D xyz\,dx\,dy\,dz$ where D is the region bounded by the positive octant of the sphere $x^2+y^2+z^2=a^2$

Solution : Let I = $\iiint\limits_D xyz\,dx\,dy\,dz$

It clear that x varies from 0 to a, y varies from 0 to $\sqrt{a^2-x^2}$ and z varies from 0 to $\sqrt{a^2-x^2-y^2}$.

Thus I = $\iiint\limits_D xyz\,dx\,dy\,dz$

$$= \int\limits_0^a \int\limits_0^{\sqrt{a^2-x^2}} \int\limits_0^{\sqrt{a^2-x^2-y^2}} xyz\,dx\,dy\,dz$$

$$= \int\limits_0^a \int\limits_0^{\sqrt{a^2-x^2}} xy\left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dx\,dy$$

$$= \frac{1}{2}\int\limits_0^a \int\limits_0^{\sqrt{a^2-x^2}} xy(a^2-x^2-y^2)\,dx\,dy$$

$$= \frac{1}{2}\int\limits_0^a \int\limits_0^{\sqrt{a^2-x^2}} (a^2xy-x^3y-xy^3)\,dx\,dy$$

$$= \frac{1}{2}\int\limits_0^a \left[a^2x\frac{y^2}{2}-x^3\frac{y^2}{2}-x\frac{y^4}{4}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{8}\int\limits_0^a \left[2a^2xy^2-2x^3y^2-xy^4\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{8}\int\limits_0^a \left[2a^2x(a^2-x^2)-2x^3(a^2-x^2)-x(a^2-x^2)^2\right]dx$$

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$$= \frac{1}{8}\int_0^a \left[2a^4x - 2a^2x^3 - 2a^2x^3 + 2x^5 - x(a^4 - 2a^2x^2 + x^4)\right]dx$$

$$= \frac{1}{8}\int_0^a \left[2a^4x - 2a^2x^3 - 2a^2x^3 + 2x^5 - a^4x + 2a^2x^3 - x^5\right]dx$$

$$= \frac{1}{8}\int_0^a \left[a^4x - 2a^2x^3 + x^5\right]dx$$

$$= \frac{1}{8}\left[a^4\frac{x^2}{2} - 2a^2\frac{x^4}{4} + \frac{x^6}{6}\right]_0^a$$

$$= \frac{1}{8}\left[a^4\frac{a^2}{2} - 2a^2\frac{a^4}{4} + \frac{a^6}{6}\right] - [0]$$

$$= \frac{1}{8}\cdot a^6\left[\frac{1}{6}\right]$$

$$= \frac{a^6}{48}$$

Hence $\iiint\limits_D xyz\,dx\,dy\,dz = \dfrac{a^6}{48}$.

Example 5.4.3 :

Evaluate $\iiint\limits_D \dfrac{dz\,dy\,dx}{(x+y+z+1)^3}$ where D is the region bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$.

Solution : Let I = $\iiint\limits_D \dfrac{dz\,dy\,dx}{(x+y+z+1)^3}$

Here the region is a tetrahedron.

Thus x varies from 0 to 1, y varies from 0 to $1-x$ and 0 to $1-x-y$

Hence I = $\displaystyle\int_0^1\int_0^{1-x}\int_0^{1-x-y} \frac{dz\,dy\,dx}{(x+y+z+1)^3}$

$$= \int_0^1\int_0^{1-x}\int_0^{1-x-y} (x+y+z+1)^{-3}\,dz\,dy\,dx$$

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$$=\int_0^1\int_0^{1-x}\left[\frac{(x+y+z+1)^{-2}}{-2}\right]_0^{1-x-y} dy\,dx$$

$$=\frac{1}{-2}\int_0^1\int_0^{1-x}\left[\left[(1+x+y+1-x-y)^{-2}\right]-(1+x+y)^{-2}\right]dy\,dx$$

$$=\frac{1}{-2}\int_0^1\int_0^{1-x}\left[\frac{1}{4}-(1+x+y)^{-2}\right]dy\,dx$$

$$=\frac{1}{-2}\int_0^1\left[\frac{1}{4}y-\frac{(1+x+y)^{-1}}{-1}\right]_0^{1-x} dy\,dx$$

$$=\frac{1}{-2}\int_0^1\left[\frac{1}{4}y+(1+x+y)^{-1}\right]_0^{1-x} dy\,dx$$

$$=\frac{1}{-2}\int_0^1\left[\frac{1}{4}(1-x)+(1+x+1-x)^{-1}-0-(1+x)^{-1}\right]dx$$

$$=\frac{1}{-2}\int_0^1\left[\frac{1}{4}(1-x)+\frac{1}{2}-\frac{1}{1+x}\right]dx$$

$$=\frac{1}{-2}\left[\frac{1}{4}\left(x-\frac{x^2}{2}\right)+\frac{1}{2}x-\log(1+x)\right]_0^1$$

$$=\frac{1}{-2}\left\{\left[\frac{1}{4}\left(1-\frac{1}{2}\right)+\frac{1}{2}-\log(1+1)\right]-0\right\}$$

$$=\frac{1}{-2}\left[\frac{1}{8}+\frac{1}{2}-\log(2)\right]$$

$$=\frac{1}{2}\log 2-\frac{5}{16}$$

Space for Hints

Check your progress

Questions :

(1) Evaluate $\iiint_D (x^2+y^2+z^2)\,dx\,dy\,dz$ where D is the region bounded by the planes $x+y+z=a$, $x=0$, $y=0$ and $z=0$.

(2) Evaluate $\iiint_D xyz(x^2+y^2+z^2)\,dx\,dy\,dz$ where D is the positive octant of the sphere $x^2+y^2+z^2=a^2$.

5.4.1 Jacobian

In certain cases by changing the given variables x,y,z to new variables, say u,v,w given by the relations $x=f(u,v,w)$, $y=g(u,v,w)$, $z=h(u,v,w)$, evaluation of multiple integrals becomes easy.

Definition : If $u=f(x,y)$ and $v=g(x,y)$ be two continuous functions of the independent variables x and y such that their first order partial derivatives are also continuous in x and y, then

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$ is called the Jacobian of u, v with respect to x, y and is denoted by

$J\left(\dfrac{u,v}{x,y}\right)$ or $\dfrac{\partial(u,v)}{\partial(x,y)}$.

(i.e) $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}.$$

Similarly, for the three variables u,v,w which are functions of x,y,z then the Jacobian of u,v,w with respect to x,y,z is given by

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Example 5.4.4 :

Find the Jacobian when cartesian coordinates are transformed into polar coordinates.

Solution : The relationship between cartesian coordinates and polar coordinates are $x = r\cos\theta$, $y = r\sin\theta$

Now $\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$

$$\therefore\ J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

Example 5.4.5 :

Find the Jacobian when cartesian coordinates are transformed into spherical coordinates.

Solution : The relationship between cartesian coordinates and spherical coordinates are $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$

Now $\frac{\partial x}{\partial r} = \sin\theta\cos\varphi$, $\frac{\partial x}{\partial \theta} = r\cos\theta\cos\varphi$, $\frac{\partial x}{\partial \varphi} = -r\sin\theta\sin\varphi$

Space for Hints

Space for Hints

$$\frac{\partial y}{\partial r} = \sin\theta\sin\varphi,\ \frac{\partial y}{\partial \theta} = r\cos\theta\sin\varphi,\ \frac{\partial y}{\partial \varphi} = r\sin\theta\cos\varphi$$

$$\frac{\partial z}{\partial r} = \cos\theta,\ \frac{\partial z}{\partial \theta} = -r\sin\theta,\ \frac{\partial y}{\partial \varphi} = 0$$

$$\therefore\ J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta\cdot\begin{vmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

Expand along C_3

$$= r^2\sin\theta\Big[-\sin\varphi\left(-\sin^2\theta\sin\varphi-\cos^2\theta\sin\varphi\right)$$

$$-\cos\varphi\left(-\sin^2\theta\cos\varphi-\cos^2\theta\cos\varphi+0\right)\Big]$$

$$= r^2\sin\theta\left[\sin^2\theta+\cos^2\theta\right]$$

$$= r^2\sin\theta$$

Thus J = $r^2\sin\theta$.

Example 5.4.5 :

Find the Jacobian when cartesian coordinates are transformed into cylindrical coordinates.

Solution : The relationship between cartesian coordinates and cylindrical coordinates are $x = r\cos\theta,\ y = r\sin\theta,\ z = z$

Now $\frac{\partial x}{\partial r} = \cos\theta,\ \frac{\partial x}{\partial \theta} = -r\sin\theta,\ \frac{\partial x}{\partial \varphi} = 0$

$$\frac{\partial y}{\partial r} = \sin\theta, \; \frac{\partial y}{\partial \theta} = r\cos\theta, \; \frac{\partial y}{\partial \varphi} = 0$$

$$\frac{\partial z}{\partial r} = 0, \; \frac{\partial z}{\partial \theta} = 0, \; \frac{\partial y}{\partial \varphi} = 1$$

$$\therefore \; J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expand along C_3

$$= 0 - 0 + 1(r\cos^2\theta + r\sin^2\theta)$$

$$= r$$

Thus $J = r$

Change of variables in double and triple integrals

The evaluation of a double or triple integral becomes more easier when we use transformation of a set variables into another set of variables.

For that we use the following theorem (without proof)

Theorem : Consider a transformation given by the equation $x = f(u,v)$ and $y = g(u,v)$ where x and y have continuous first order partial derivatives. Let the region D in the xy plane be mapped into the region D^* in the uv plane. Further we assume that the Jacobian transformation $J \neq 0$ for all points in D. Then $\iint\limits_D f(x,y)\,dx\,dy = \iint\limits_D \varphi(u,v)\,|\,J\,|\,du\,dv$.

Similarly for triple integrals,

$$\iiint\limits_D f(x,y,z)\,dx\,dy\,dz = \iiint\limits_D \varphi(u,v,w)\,|\,J\,|\,du\,dv\,dw\,.$$

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Example 5.4.6 :

Evaluate $\iint (x^2+y^2)^{7/2}\,dx\,dy$ over a circle $x^2+y^2=1$.

Solution : Let I = $\iint (x^2+y^2)^{7/2}\,dx\,dy$

The polar coordinates of the circle $x^2+y^2=1$ are $x=r\cos\theta$, $y=r\sin\theta$ where $0\le r\le 1$ and $0\le\theta\le 2\pi$.

Now $x=r\cos\theta$, $y=r\sin\theta$

$$\therefore \frac{\partial x}{\partial r}=\cos\theta,\ \frac{\partial x}{\partial\theta}=-r\sin\theta$$

$$\frac{\partial y}{\partial r}=\sin\theta,\ \frac{\partial y}{\partial\theta}=r\cos\theta$$

$$\text{Thus J} = \begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$$

$$= \begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}$$

$$= r(\cos^2\theta+\sin^2\theta)$$

$$= r$$

$$\therefore \text{I} = \iint (x^2+y^2)^{7/2}\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 (r^2)^{7/2}\,|J|\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^7\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^8\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^9}{9}\right]_0^1 d\theta$$

$$= \int_0^{2\pi}\left[\frac{1}{9}-0\right]d\theta$$

$$= \frac{1}{9}\int_0^{2\pi} d\theta$$

$$= \frac{1}{9}\left[\theta\right]_0^{2\pi}$$

$$= \frac{2\pi}{9}$$

Hence $I = \dfrac{2\pi}{9}$

Example 5.4.7 :

Evaluate $\iint xy(x^2+y^2)^{3/2}\,dx\,dy$ over the positive quadrant of the circle $x^2+y^2=1$.

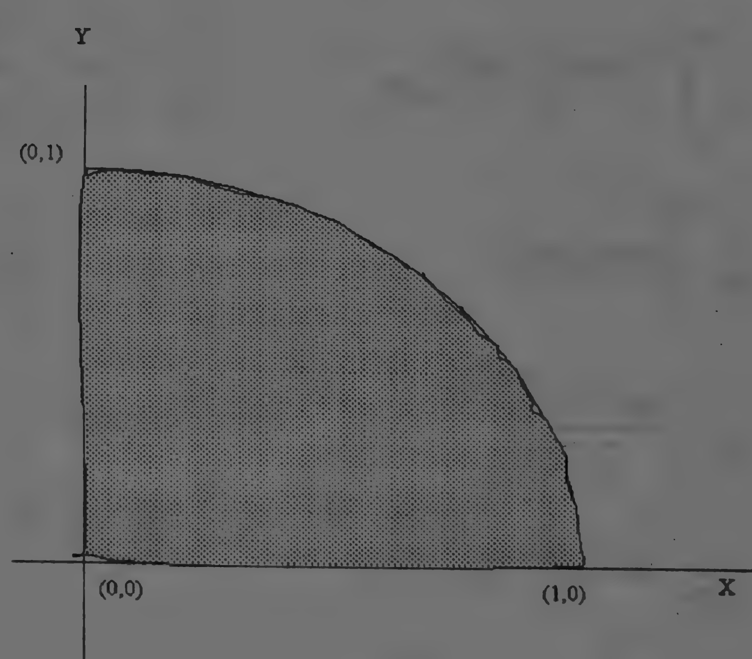
Solution : Let $I = \iint xy(x^2+y^2)^{3/2}\,dx\,dy$

The polar coordinates of the circle $x^2+y^2=1$ are $x=r\cos\theta$, $y=r\sin\theta$ where $0\le r\le 1$ and $0\le\theta\le 2\pi$.

Now $x=r\cos\theta$, $y=r\sin\theta$

$$\therefore \frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$



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$$\text{Thus } J = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

$$\therefore\ I = \iint xy(x^2+y^2)^{3/2}\,dx\,dy$$

$$= \int_0^{\pi/2}\int_0^1 r\cos\theta\, r\sin\theta\,(r^2)^{3/2}\,|J|\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 r^5\cos\theta\, r\sin\theta\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 r^6\cos\theta\, r\sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\cos\theta\, r\sin\theta\left[\frac{r^7}{7}\right]_0^1 d\theta$$

$$= \int_0^{\pi/2}\cos\theta\, r\sin\theta\left[\frac{1}{7}-0\right]d\theta$$

$$= \frac{1}{7}\int_0^{\pi/2}\frac{2\cos\theta\, r\sin\theta}{2}\,d\theta$$

$$= \frac{1}{14}\int_0^{\pi/2}\sin 2\theta\,d\theta$$

$$= \frac{1}{14}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= -\frac{1}{28}\left[-1-1\right]$$

$$= \frac{1}{14}$$

Thus I = $\frac{1}{14}$

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Example 5.4.8 :

Find the volume of the cylinder with base radius a and height h.

Solution :

The equation of the cylinder with base radius a and height h is given by $x^2 + y^2 \le a; 0 \le z \le h$.

Here $0 \le r \le a$, $0 \le \theta \le \pi$, $0 \le z \le h$

Thus the volume of the cylinder is V = $\iiint\limits_D dx\,dy\,dz$.

We know that the cylindrical coordinates are $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

Now $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$\therefore \quad \frac{\partial x}{\partial r} = \cos\theta, \ \frac{\partial x}{\partial \theta} = -r\sin\theta, \ \frac{\partial x}{\partial \varphi} = 0$$

$$\frac{\partial y}{\partial r} = \sin\theta, \ \frac{\partial y}{\partial \theta} = r\cos\theta, \ \frac{\partial y}{\partial \varphi} = 0$$

$$\frac{\partial z}{\partial r} = 0, \ \frac{\partial z}{\partial \theta} = 0, \ \frac{\partial y}{\partial \varphi} = 1$$

$$\therefore \ J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expand along C_3

$$= 0 - 0 + 1(r\cos^2\theta + r\sin^2\theta)$$

$$= r$$

Thus J = r

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$$\therefore V = \iiint_D dx\,dy\,dz$$

$$= \int_0^h \int_0^{2\pi} \int_0^a |J|\,dr\,d\theta\,dz$$

$$= \int_0^h \int_0^{2\pi} \int_0^a r\,dr\,d\theta\,dz$$

$$= [z]_0^h \cdot [\theta]_0^{2\pi} \cdot \left[\frac{r^2}{2}\right]_0^a$$

$$= [h-0]\cdot[2\pi-0]\cdot\left[\frac{a^2}{2}-0\right]$$

$$= h\cdot 2\pi\cdot\frac{a^2}{2}$$

$$= \pi a^2 h$$

Hence volume of the cylinder is $\pi a^2 h$ cubic units.

Example 5.4.9 :

Find the volume of the sphere of radius a.

Solution : The equation of the sphere of radius a is given by $x^2+y^2+z^2=a^2$.

We know that the relationship between cartesian coordinates and spherical coordinates are $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$

Here $0 \le r \le a$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$

Now $\frac{\partial x}{\partial r} = \sin\theta\cos\varphi$, $\frac{\partial x}{\partial \theta} = r\cos\theta\cos\varphi$, $\frac{\partial x}{\partial \varphi} = -r\sin\theta\sin\varphi$,

$\frac{\partial y}{\partial r} = \sin\theta\sin\varphi$, $\frac{\partial y}{\partial \theta} = r\cos\theta\sin\varphi$, $\frac{\partial y}{\partial \varphi} = r\sin\theta\cos\varphi$,

$\frac{\partial z}{\partial r} = \cos\theta$, $\frac{\partial z}{\partial \theta} = -r\sin\theta$, $\frac{\partial y}{\partial \varphi} = 0$.

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$$\therefore\ J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta \cdot \begin{vmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

Expand along C_3

$$= r^2\sin\theta\Big[-\sin\varphi\left(-\sin^2\theta\sin\varphi - \cos^2\theta\sin\varphi\right)$$

$$-\cos\varphi\left(-\sin^2\theta\cos\varphi - \cos^2\theta\cos\varphi + 0\right)\Big]$$

$$= r^2\sin\theta\left[\sin^2\theta + \cos^2\theta\right]$$

$$= r^2\sin\theta$$

hus J = $r^2\sin\theta$.

. Volume of the sphere = $\iiint\limits_V dx\,dy\,dz$

$$= \int_0^a\int_0^\pi\int_0^{2\pi} r^2\sin\theta\,dr\,d\theta\,d\varphi$$

$$= \left[\frac{r^3}{3}\right]_0^a \cdot \left[-\cos\theta\right]_0^\pi \cdot \left[\varphi\right]_0^{2\pi}$$

$$= \left[\frac{a^3}{3} - 0\right] \cdot \left[1+1\right] \cdot \left[2\pi - 0\right]$$

$$= \left[\frac{a^3}{3}\right] \cdot \left[2\right] \cdot \left[2\pi\right]$$

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$$= \frac{4}{3}\pi a^3$$

Thus the volume of the sphere of radius a is $\frac{4}{3}\pi a^3$ cubic units.

Example 5.4.10 :

Given that $x+y=u$, $y=uv$, change the variables to u,v in the integral $\iint xy\sqrt{1-x-y}\,dx\,dy$ taken over the area of the triangle with sides $x=0$, $y=0$, $x+y=1$ and evaluate it.

Solution : Let I = $\iint xy\sqrt{1-x-y}\,dx\,dy$

Given that $x+y=u$, $y=uv$

$\therefore\ x=u-uv,\ \ y=uv$

Now $\frac{\partial x}{\partial u}=1-v$, $\frac{\partial x}{\partial v}=-u$,

$$\frac{\partial y}{\partial u}=v,\ \ \frac{\partial y}{\partial v}=u$$

$$\therefore\ J=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$$

$$=\begin{vmatrix}1-v & -u\\ v & u\end{vmatrix}$$

$$=(1-v)u+uv$$

$$=u$$

The area of the triangle with sides $x=0$, $y=0$, $x+y=1$ transforms into the area of the square with sides $u=0$, $u=1$, $v=0$, $v=1$

$$\therefore\ I=\iint xy\sqrt{1-x-y}\,dx\,dy$$

$$=\int_0^1\int_0^1\sqrt{u^2v(1-v)(1-u)}\,|J|\,du\,dv$$

$$=\int_0^1\int_0^1\sqrt{u^2v(1-v)(1-u)}\,u\,du\,dv$$

$$= \int_0^1 \int_0^1 u^2 \sqrt{1-u} \sqrt{v(1-v)}\, du\, dv$$

$$= \left[\int_0^1 u^2 \sqrt{1-u}\, du \right] \cdot \left[\int_0^1 \sqrt{v} \sqrt{1-v}\, dv \right]$$

$$= \frac{2\pi}{105}$$

Hence $I = \dfrac{2\pi}{105}$.

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Summary

In this unit we have leaned reduction formula for integration, Bernoulli's formula, double and triple integrations.

Further Reading

You can also refer the following books for further reading.

(1) Calculus by Arumugam and Isaac

(2) Differential Calculus by Shanti Narayanan

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UNIT VI

RELATIONS BETWEEN ROOTS AND COEFFICIENTS

Introduction

Unit Objectives

Unit Structure

Check your progress

Summary

Further Reading

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Objectives :

In this unit, we are going to discuss to find the imaginary and rational roots of an equation, relationship between roots and the coefficients and symmetric functions of roots of an equation.

After completing this unit, students may able to know

- To find imaginary roots
- To find rational roots
- Relationship between roots and the coefficients
- Symmetric function of roots

Introduction

Theory of equations is that branch of Mathematics, which deals with the solutions of equations. A polynomial with real coefficients is of the form

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n \text{ ------ (6.1)}$$

where $a_0 \neq 0$, n is a non-negative integer and x is a variable which may assume real or complex values.

Note (1) :

In the above polynomial (6.1) n is called *degree of the polynomial*.

Note (2) :

A polynomial of degree 1 is called linear

A polynomial of degree 2 is called quadratic

A polynomial of degree 3 is called cubic

A polynomial of degree 4 is called biquadratic or quartic

A polynomial of degree 5 is called quantic

A polynomial of degree 6 is called sextic

A polynomial of degree 7 is called quintic

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Definition : A polynomial $f(x)$ is equated to zero is called an algebraic equation.

Definition : Let $f(x)$ be a polynomial of degree n and $\alpha \in \mathbf{R}$ or $\alpha \in \mathbf{C}$. If $f(\alpha)=0$ then α is called a root of $f(x)$.

Theorem 6.1.1 : If $f(x)$ is a polynomial, then $f(a)$ is the remainder when $f(x)$ is divided by $x-a$.

Proof : Let $Q(x)$ and R be the quotient and remainder when $f(x)$ is divided by $x-a$.

Then $f(x) = (x-a)Q(x)+R$ ---------------------------------- (6.2)

Put $x=a$ in (6.2), we get $f(a)=R$

Thus $f(a)$ is the remainder when $f(x)$ is divided by $x-a$.

This proves the theorem.

Note :

If a is a root of the polynomial $f(x)$ then $x-a$ is a factor of the polynomial.

Theorem 6.1.2 :

If $f(a)$ and $f(b)$ are of different signs, then at least one root of the equation $f(x)=0$ must lie between a and b.

Proof : As x changes gradually from a to b, then the function $f(x)$ changes gradually from $f(a)$ to $f(b)$ and therefore must pass through all intermediate values.

Since $f(a)$ and $f(b)$ have different sign, then zero must lies between $f(a)$ and $f(b)$.

(i.e) $f(x)$ assumes the value zero for at least for one value of x between a and b.

This proves the theorem.

Note (1) :

If $f(a)$ and $f(b)$ have like signs, an even number of roots of $f(x)=0$ lie between a and b or else there is no root between a and b.

Note (2) :

If $f(a)$ and $f(b)$ have different signs, an odd number of roots of $f(x)=0$, lie between a and b.

Theorem 6.1.3 :

If $f(x)=0$ is an equation of odd degree, it has at least one real root whose sign is opposite to that of the last term.

Proof : Let $f(x)= x^n + p_1 x^{n-1} + p_2 x^{n-2} + \text{L} + p_n$ ----- (6.3)

be polynomial of degree n.

put $x=-\infty,\ 0,\ \infty$ in (6.3), we have,

$f(-\infty)=-\infty$ since n is odd,

$f(0)=p_n$,

$f(+\infty)=+\infty$

Thus if $p_n>0$, then $f(x)=0$ has at least one root lying between $-\infty$ and 0;

and if $p_n<0$, then $f(x)=0$ has at least one root lying between 0 and $+\infty$.

Hence if $f(x)=0$ is an equation of odd degree, it has at least one real root whose sign is opposite to that of the last term.

This proves the theorem.

Theorem 6.1.4 :

Every n^{th} degree equation $f(x)=0$ has exactly n roots.

Proof : We shall prove the result using induction on n.

If $n=1$ then $f(x)=a_0x+a_1$.

Then $f(x)=0$

$\Rightarrow a_0x+a_1=0$

$$\Rightarrow x=-\frac{a_1}{a_0}$$

(i.e) $-\dfrac{a_1}{a_0}$ is a root of $f(x)=0$.

Thus, the result is true for $n=1$.

Make an induction hypothesis as the result is true for all polynomials of degree less than n.

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Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \text{L} + a_n$ be a polynomial of degree *n*.

Let α be a root of $f(x) = 0$ with multiplicity *r*.

(i.e) $f(x) = (x-\alpha)^r g(x)$ -------------------------------------- (6.4)

where $g(\alpha) \neq 0$.

Clearly degree of $g(x)$ is $n-1<n$

By induction hypothesis $g(x)$ has exactly $n-1$ roots.

Thus $f(x)$ has $n-1+1=n$ roots.

$\therefore$ By induction hypothesis n^{th} degree polynomial has exactly *n* roots.

This proves the theorem.

Example 6.1.1 :

If α be a real root of the cubic equation $x^3 + px^2 + qx + r = 0$ of which the coefficients are real, show that the other two roots are real if $p^2 \geq 4q + 2p\alpha + 3\alpha^2$.

Proof : Given that α is a real root of $x^3 + px^2 + qx + r = 0$.

$\therefore$ $x-\alpha$ is a root of $x^3 + px^2 + qx + r = 0$

Let $x^3 + px^2 + qx + r = (x-\alpha)(x^2 + ax + b)$

$$= x^3 + ax^2 + bx - \alpha x^2 - a\alpha x - b\alpha$$

$$= x^3 + (a-\alpha)x^2 + (b - a\alpha)x - b\alpha$$

Equating the coefficients of like powers of *x* on both sides, we get,

$p = a - \alpha$,

$q = b - a\alpha$,

$r = -b\alpha$

$\therefore$ $a = p + \alpha$,

$b = q + a\alpha$

$= q + \alpha(p + \alpha)$

$= q + p\alpha + \alpha^2$

Hence the other two roots of the equation are the roots of

$$x^2+(p+\alpha)x+(q+p\alpha+\alpha^2)=0 \quad \text{-----------------------------} (6.5)$$

The roots of (6.5) are real if $\Delta \geq 0$

(i.e) $(p+\alpha)^2-4(q+p\alpha+\alpha^2)\geq 0$

(i.e) $p^2+2p\alpha+\alpha^2-4q-4p\alpha-4\alpha^2\geq 0$

(i.e) $p^2-2p\alpha-4q-3\alpha^2\geq 0$

(i.e) $p^2\geq 4q+2p\alpha+3\alpha^2$

This proves the problem.

Example 6.1.2 :

Show that if a,b,c are real, the roots of $\dfrac{1}{x+a}+\dfrac{1}{x+b}+\dfrac{1}{x+c}=\dfrac{3}{x}$ are real

Proof : Given that $\dfrac{1}{x+a}+\dfrac{1}{x+b}+\dfrac{1}{x+c}=\dfrac{3}{x}$.

(i.e) $x(x+b)(x+c)+x(x+a)(x+c)+x(x+a)(x+b)=3(x+a)(x+b)(x+c)$

Let $f(x)=x(x+b)(x+c)+x(x+a)(x+c)+x(x+a)(x+b)$

$$-3(x+a)(x+b)(x+c)$$

Clearly degree of $f(x)=2$

$\therefore\ f(-a)=-a(b-a)(c-a)$,

$f(-b)=-b(c-b)(a-b)$

$f(-c)=-c(a-c)(b-c)$

Without loss of generality, assume that $a>b>c>0$

Then $a-b>0$, $b-a>0$, $a-c>0$

$\therefore f(-a)<0$, $f(-b)>0$, $f(-c)<0$.

Thus the equation has at least one real root between $-a$ and $-b$; another root between $-b$ and $-c$.

Since degree of $f(x)=2$ then $f(x)=0$ has exactly two roots and that two roots are real.

This proves the problem.

Theorem 6.1.5 :

In an equation with real coefficients, imaginary roots occur in pairs.

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Proof : Let $\alpha+i\beta$ be an imaginary root of the equation $f(x)=0$.

Claim : $\alpha-i\beta$ is a root of $f(x)=0$.

Now $\alpha+i\beta$ is a root of $f(x)=0$.

$\therefore\ f(\alpha+i\beta)=0$ -- (6.6)

Now $\left[x-(\alpha+i\beta)\right]\cdot\left[x-(\alpha-i\beta)\right]$

$$=\left[(x-\alpha)-i\beta\right]\cdot\left[(x-\alpha)+i\beta\right]$$

$$=(x-\alpha)^2+\beta^2$$

Let $Q(x)$ and $Rx+R_1$ be the quotient and remainder when $f(x)$ is divided by $(x-\alpha)^2+\beta^2$.

(i.e) $f(x)=\left[(x-\alpha)^2+\beta^2\right]Q(x)+Rx+R_1$ ---------------- (6.7)

Put $x=\alpha+i\beta$ in (6.7), we get,

$$f(\alpha+i\beta)=\left[(\alpha+i\beta-\alpha)^2+\beta^2\right]Q(\alpha+i\beta)+R(\alpha+i\beta)+R_1$$

(i.e) $0=0+(R\alpha+R_1)+iR\beta$ ------------------------------------- (6.8)

Equating real and imaginary parts of (6.8), we get,

$R\alpha+R_1=0$ -- (6.9)

$R\beta=0$ --- (6.10)

Since $\beta\neq0$, (6.10) $\Rightarrow$ $R=0$

From (6.9), $(0)\alpha+R_1=0\Rightarrow R_1=0$

$\therefore$ (6.7) $\Rightarrow$ $f(x)=\left[(x-\alpha)^2+\beta^2\right]Q(x)$ ------------------ (6.11)

Put $x=\alpha-i\beta$ in (6.11), we get,

$$f(\alpha-i\beta)=\left[(\alpha-i\beta-\alpha)^2+\beta^2\right]Q(\alpha-i\beta)$$

$\Rightarrow\ f(\alpha-i\beta)=0$

$\Rightarrow\ \alpha-i\beta$ is a root of $f(x)=0$.

Hence in an equation with real coefficients, imaginary roots occur in pairs.

This proves theorem.

Example 6.1.3 :

Find the equation with rational coefficients whose roots are $1+5\sqrt{-1}$, $5-\sqrt{-1}$.

Solution : Given that $1+5i$, $5-i$ are the roots of an equation.

$\therefore$ the other two roots are $1-5i$ and $5+i$

Thus the required equation is

$$\left(x-(1+5i)\right)\left(x-(1-5i)\right)\left(x-(5-i)\right)\left(x-(5+i)\right)=0$$

(i.e) $\left((x-1)-5i\right)\left((x-1)+5i\right)\left((x-5)+i\right)\left((x-5)-i\right)=0$

(i.e) $\left((x-1)^2+25\right)\left((x-5)^2+1\right)=0$

(i.e) $(x^2-2x+1+25)(x^2-10x+25+1)=0$

(i.e) $(x^2-2x+26)(x^2-10x+26)=0$

(i.e) $x^4-10x^3+26x^2-2x^3+20x^2-52x+26x^2-260x+676=0$

(i.e) $x^4-12x^3+72x^2-312x+676=0$

which is the required equation.

Example 6.1.4 :

Solve the equation $x^4+2x^3-5x^2+6x+2=0$ given that $1+\sqrt{-1}$ is a root.

Solution : Given that $1+\sqrt{-1}=1+i$ is a root of an equation.

$x^4+2x^3-5x^2+6x+2=0$ -- (6.12)

$\therefore$ $1-i$ is also a root of the equation (6.12)

Thus $x-(1+i)$, $x-(1-i)$ are factors of the equation (6.12).

(i.e) $\left(x-(1+i)\right)\left(x-(1-i)\right)$ is factors of the equation (6.12)

(i.e) x^2-2x+2 is factors of the equation (6.12)

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$$
\begin{array}{r|l}
 & x^2+4x+1 \\
\hline
x^2-2x+2 & x^4+2x^3-5x^2+6x+2 \\
 & x^4-2x^3+2x^2 \\
\hline
 & 4x^3-7x^2+6x \\
 & 4x^3-8x^2+8x \\
\hline
 & x^2-2x+2 \\
 & x^2-2x+2 \\
\hline
 & \quad 0
\end{array}
$$

$\therefore\ x^4+2x^3-5x^2+6x+2 = (x^2-2x+2)(x^2+4x+1)$.

Thus $x^4+2x^3-5x^2+6x+2=0$

$\Rightarrow\ (x^2-2x+2)(x^2+4x+1)=0$

$\Rightarrow\ x^2-2x+2=0$ or $x^2+4x+1=0$

Now $x^2+4x+1=0$

$$\Rightarrow\ x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow\ x=\frac{-4\pm\sqrt{16-4}}{2}$$

$$\Rightarrow\ x=\frac{-4\pm\sqrt{12}}{2}$$

$$\Rightarrow\ x=-2\pm\sqrt{3}$$

Hence the roots of the equation are $1\pm i$, $-2\pm\sqrt{3}$.

Check your progress

Questions :

(1) Solve $x^4-4x^2+8x+35=0$ given that $2+i\sqrt{3}$ is a root of it.

(2) Solve $3x^3-4x^2+x+88=0$ which has a root $2-i\sqrt{7}$.

(Answer : (1) $2\pm i\sqrt{3}, -2\pm i$ and (2) $2\pm i\sqrt{7}$, $-\frac{8}{3}$)

6.2. Rational roots

Space for Hints

Theorem 6.2.1 :

In an equation with rational coefficients, irrational roots occur in pairs.

Proof : Let $a+\sqrt{b}$ be an imaginary root of the equation $f(x)=0$.

Claim : $a-\sqrt{b}$ is a root of $f(x)=0$.

Now $a+\sqrt{b}$ is a root of $f(x)=0$.

$\therefore\ f(a+\sqrt{b})=0$ -- (6.13)

Now $\left[x-(a+\sqrt{b})\right]\cdot\left[x-(a-\sqrt{b})\right]$

$=\left[(x-a)-\sqrt{b}\right]\cdot\left[(x-a)+\sqrt{b}\right]$

$=(x-a)^2-b$

Let $Q(x)$ and $Rx+R_1$ be the quotient and remainder when $f(x)$ is divided by $(x-a)^2-b$.

(i.e) $f(x)=\left[(x-a)^2-b\right]Q(x)+Rx+R_1$ -------------------- (6.14)

Put $x=a+\sqrt{b}$ in (6.14), we get,

$f(a+\sqrt{b})=\left[(a+\sqrt{b}-a)^2-b\right]Q(a+\sqrt{b})+R(a+\sqrt{b})+R_1$

(i.e) $0=0+(Ra+R_1)+R\sqrt{b}$ ------------------------------------ (6.15)

Equating rational and irrational parts of (6.15), we get,

$Ra+R_1=0$ -- (6.16)

$R=0$ -- (6.17)

From (6.16) and (6.17), $(0)a+R_1=0 \Rightarrow R_1=0$

$\therefore$ (6.14) $\Rightarrow\ f(x)=\left[(x-a)^2-b\right]Q(x)$ --------------------- (6.18)

Put $x=a-\sqrt{b}$ in (6.18), we get,

$f(a-\sqrt{b})=\left[(a-\sqrt{b}-a)^2-b\right]Q(a-\sqrt{b})$

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$\Rightarrow f(a-\sqrt{b}) = 0$

$\Rightarrow a-\sqrt{b}$ is a root of $f(x)=0$.

Hence in an equation with rational coefficients, irrational roots occur in pairs. This proves theorem.

Example 6.2.1 :

Solve the equation $6x^4-13x^3-35x^2-x+3=0$ given that $2-\sqrt{3}$ is a root.

Solution : Given that $2-\sqrt{3}$ is a root of an equation.

$6x^4-13x^3-35x^2-x+3=0$ ---------------------------------- (6.19)

$\therefore\ 2+\sqrt{3}$ is also a root of the equation (6.19)

Thus $x-(2+\sqrt{3})$, $x-(2-\sqrt{3})$ are factors of the equation (6.19).

(i.e) $\left(x-(2+\sqrt{3})\right)\left(x-(2-\sqrt{3})\right)$ is factors of the equation (6.19)

(i e) x^2-4x+1 is factors of the equation (6.19)

$$
\begin{array}{r|l}
 & 6x^2+11x+3 \\
\hline
x^2-4x+1 & 6x^4-13x^3-35x^2-x+3 \\
 & 6x^4-24x^3+6x^2 \\
\hline
 & 11x^3-41x^2-x \\
 & 11x^3-44x^2+11x \\
\hline
 & 3x^2-12x+3 \\
 & 3x^2-12x+3 \\
\hline
 & 0
\end{array}
$$

$\therefore\ 6x^4-13x^3-35x^2-x+3 = (x^2-4x+1)\,(6x^2+11x+3)$

Thus $x^4+2x^3-5x^2+6x+2=0$

$\Rightarrow (x^2-4x+1)\,(6x^2+11x+3)=0$

$\Rightarrow x^2-4x+1=0$ or $6x^2+11x+3=0$

Now $6x^2+11x+3=0$

$$\Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow x=\frac{-114\pm\sqrt{121-4(6)(3)}}{2(6)}$$

$$\Rightarrow x=\frac{-4\pm\sqrt{49}}{12}$$

$$\Rightarrow x=-\frac{1}{3} \text{ or } x=-\frac{3}{2}$$

Hence the roots of the equation are $2\pm\sqrt{3}$, $-\frac{1}{3}$, $-\frac{3}{2}$.

Check your progress

Questions :

(1) Solve $x^4-6x^3+11x^2-10x+2=0$ given that $2+\sqrt{3}$ is a root of it.

(2) Solve $x^4-14x^3+46x^2-42x+9=0$ which has a root $5-\sqrt{22}$.

(Answer : (2) $5\pm\sqrt{22}$, 3, 1)

Example 6.2.2 :

Solve the equation $2x^6-3x^5+5x^4+6x^3-27x+81=0$ given that $\sqrt{2}-\sqrt{-1}$ is a root.

Solution : Given that $\sqrt{2}-\sqrt{-1}$ is a root of an equation.

$2x^6-3x^5+5x^4+6x^3-27x+81=0$ ------------------------- (6.20)

$\therefore$ $-\sqrt{2}-i$, $\sqrt{2}+i$, $-\sqrt{2}-i$ are roots of the equation (6.20)

Thus $x-(\sqrt{2}-i)$, $x-(\sqrt{2}+i)$, $x-(-\sqrt{2}-i)$, $x-(-\sqrt{2}+i)$ are factors of the equation (6.20).

(i.e) $\left(x-(\sqrt{2}-i)\right)\left(x-(\sqrt{2}+i)\right)\left(x-(\sqrt{2}+i)\right)\left(x-(-\sqrt{2}+i)\right)$ is factor of the equation (6.20)

Now $\left(x-(\sqrt{2}-i)\right)\left(x-(\sqrt{2}+i)\right)\left(x-(\sqrt{2}+i)\right)\left(x-(-\sqrt{2}+i)\right)$

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$$= \left((x-\sqrt{2})+i\right)\left((x-\sqrt{2})-i\right)\left((x+\sqrt{2})-i\right)\left((x+\sqrt{2})+i\right)$$

$$= \left(\left(x-\sqrt{2}\right)^2+1\right)\left(\left(x+\sqrt{2}\right)^2+1\right)$$

$$= \left(x^2+3\right)^2-\left(2\sqrt{2}x\right)^2$$

$$= x^4-2x^2+9$$

(i.e) x^4-2x^2+9 is factors of the equation (6.20)

$$\begin{array}{r|l} & 2x^2-3x+9 \\ \hline x^4-2x^2+9 & 2x^6-3x^5+5x^4+6x^3-27x+81 \\ & 2x^6-4x^4+18x^2 \\ \hline & -3x^5+9x^4+6x^3-18x^2-27x \\ & -3x^5+6x^3-27x \\ \hline & 9x^4-18x^2+81 \\ & 9x^4-18x^2+81 \\ \hline & 0 \end{array}$$

$\therefore\ 2x^6-3x^5+5x^4+6x^3-27x+81 = (x^4-2x^2+9)\ (2x^2-3x+9)$

Thus $2x^6-3x^5+5x^4+6x^3-27x+81=0$

$\Rightarrow\ (x^4-2x^2+9)\ (2x^2-3x+9)=0$

$\Rightarrow\ x^4-2x^2+9=0$ or $2x^2-3x+9=0$

Now $2x^2-3x+9=0$

$$\Rightarrow\ x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow\ x=\frac{3\pm\sqrt{9-72}}{4}$$

$$\Rightarrow\ x=\frac{3\pm i3\sqrt{7}}{4}$$

$$\Rightarrow\ x = \frac{3}{4}(1 \pm i\sqrt{7})$$

Hence the roots of the equation are $\pm\sqrt{2} \pm i$, $\frac{3}{4}(1 \pm i\sqrt{7})$.

Check your progress

Questions :

Solve the equation $x^6 - 4x^5 - 11x^4 + 40x^3 + 11x^2 - 4x - 1 = 0$ given that $\sqrt{2} - \sqrt{3}$ is a root.

6.3. Relations between roots and coefficients of equation

In this section we shall find the relations between the roots and the co-efficients of the equation.

Let $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$ be the roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_n = 0$$

Now $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_n$

$$= (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n)$$

$$= x^n - \left(\sum \alpha_1\right)x^{n-1} + \left(\sum \alpha_1\alpha_2\right)x^{n-2} + \cdots + (-1)^n \alpha_1\alpha_2\alpha_3\cdots\alpha_n$$

$$= x^n - S_1 x^{n-1} + S_2 x^{n-2} + \cdots + (-1)^n S_n$$ where S_r is the sum of products of the roots of the equation taken r at a time.

Equating the coefficients of the like powers on both sides, we have,

$$-p_1 = S_1,$$

$$(-1)^2 p_2 = S_2,$$

$$(-1)^3 p_3 = S_3,$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$(-1)^n p_n = S_n.$$

which is the required relationship between the roots and coefficients of the equation.

Space for Hints

Space for Hints

Note :

If $\alpha_1, \alpha_2, \alpha_3, \mathrm{L}\ , \alpha_n$ are the roots of the equation

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \mathrm{L}\ + a_n = 0 \text{ where } a_0 \neq 0.$$

Then the above equation can be rewritten as

$$x^n + \frac{a_1}{a_0} x^{n-1} + \frac{a_2}{a_0} x^{n-2} + \mathrm{L}\ + \frac{a_n}{a_0} = 0.$$

Thus $\quad \sum \alpha_1 = -\frac{a_1}{a_0},$

$$\sum \alpha_1 \alpha_2 = \frac{a_2}{a_0}$$

$$\mathrm{M} \qquad \mathrm{M} \qquad \mathrm{M}$$

$$\alpha_1 \alpha_2 \alpha_3 \mathrm{L}\ \alpha_n = (-1)^n \frac{a_n}{a_0}.$$

Example 6.3.1 :

Show that the roots of the equation $x^3 + px^2 + qx + r = 0$ are in arithmetic progression if $p^3 - 9pq + 27r = 0$

Proof : Let the roots of $x^3 + px^2 + qx + r = 0$ ---------------- (6.21)

be $\alpha - \delta, \alpha, \alpha + \delta$.

$\therefore \quad (\alpha - \delta) + \alpha + (\alpha + \delta) = -p$

$(\alpha - \delta)\alpha + (\alpha - \delta)(\alpha + \delta) + \alpha(\alpha + \delta) = q$

$(\alpha - \delta)\alpha(\alpha + \delta) = -r$

Simplifying the above equations, we get,

$3\alpha = -p$ -- (6.22)

$3\alpha^2 - \delta^2 = q$ -- (6.23)

$\alpha^3 - \alpha\delta^2 = -r$ --- (6.24)

From (6.22), we get, $\alpha = -\frac{p}{3}$ ------------------------------------- (6.25)

From (6.23) and (6.25), we have, $3\alpha^2 - \delta^2 = q$

$\Rightarrow \delta^2 = 3\alpha^2 - q$

$$\Rightarrow \delta^2 = 3\left(-\frac{p}{3}\right)^2 - q$$

$$\Rightarrow \delta^2 = \frac{p^2}{3} - q \text{ -- (6.26)}$$

From (6.24), (6.25) and (6.26), we have,

$$\left(-\frac{p}{3}\right)^3 - \left(-\frac{p}{3}\right)\left(\frac{p^2}{3} - q\right) = -r$$

(i.e) $\dfrac{p^3}{27} - \dfrac{p^3}{9} + \dfrac{pq}{3} = -r$

(i.e) $p^3 - 3p^3 + 9pq = -27r$

(i.e) $-2p^3 + 9pq = -27r$

(i.e) $2p^3 - 9pq + 27r = 0$

This proves the problem.

Example 6.3.2 :

Find the condition that the roots of the equation $ax^3 + 3bx^2 + 3cx + d = 0$ may be in geometric progression.

Solution : Let the roots of $ax^3 + 3bx^2 + 3cx + d = 0$ -------- (6.27)

be $\dfrac{\alpha}{\delta}, \alpha, \alpha\delta$

$$\therefore \quad \frac{\alpha}{\delta} + \alpha + \alpha\delta = -\frac{3b}{a} \text{ ------------------------------------- (6.28)}$$

$$\frac{\alpha}{\delta}\alpha + \frac{\alpha}{\delta}\alpha\delta + \alpha\,\alpha\delta = \frac{3c}{a} \text{ ---------------------------------- (6.29)}$$

$$\frac{\alpha}{\delta}\alpha\,\alpha\delta = -\frac{d}{a} \quad (6.30)$$

From (6.28), we have, $\alpha\left(\dfrac{1}{\delta} + 1 + \delta\right) = -\dfrac{3b}{a}$ ----------------- (6.31)

Form (6.29), we have, $\alpha^2\left(\dfrac{1}{\delta} + 1 + \delta\right) = \dfrac{3c}{a}$ ----------------- (6.32)

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Divide (6.32) by (6.31), we get, $\alpha = -\frac{c}{b}$

Substituting the value of α in (6.30), we get,

$$\left(-\frac{c}{b}\right)^3 = -\frac{d}{a}$$

$$-\frac{c^3}{b^3} = -\frac{d}{a}$$

$$ac^3 = b^3 d.$$

Example 6.3.3 :

If the sum of two roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ equals the sum of the other two, prove that $p^3 + 8r = 4pq$.

Proof : Let $\alpha, \beta, \gamma, \delta$ be the roots of $x^4 + px^3 + qx^2 + rx + s = 0$

Let $\alpha + \beta = \gamma + \delta$ -- (6.33)

From the relations of the coefficients and the roots, we have,

$\alpha + \beta + \gamma + \delta = -p$ -- (6.34)

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = q$ ------------------------ (6.35)

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -r$ --------------------------- (6.36)

$\alpha\beta\gamma\delta = s$ -- (6.37)

From (6.33) and (6.34), we have, $\alpha + \beta + \alpha + \beta = -p$

(i.e) $2(\alpha + \beta) = -p$

(i.e) $\alpha + \beta = -\frac{p}{2}$ -- (6.38)

Now (6.35) can be rewritten as $\alpha\beta + \gamma\delta + (\alpha + \beta)(\gamma + \delta) = q$

(i.e) $\alpha\beta + \gamma\delta + \left(-\frac{p}{2}\right)\left(-\frac{p}{2}\right) = q$

(i.e) $\alpha\beta + \gamma\delta = q - \frac{p^2}{4}$ -- (6.39)

Again (6.36) can be written as $\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -r$

$\alpha\beta(\alpha + \beta) + \gamma\delta(\alpha + \beta) = -r$

$$(\alpha+\beta)(\alpha\beta+\gamma\delta) = -r$$

$$\left(-\frac{p}{2}\right)(\alpha\beta+\gamma\delta) = -r$$

$$\alpha\beta+\gamma\delta = \frac{2r}{p} \quad \text{-----} \quad (6.40)$$

From (6.39) and (6.40), we have,

$$q-\frac{p^2}{4} = \frac{2r}{p}$$

(i.e) $4pq-p^3=8r$

(i.e) $p^3+8r=4pq$ which is the required condition.

Example 6.3.4 :

Find the condition that the general biquadratic equation $ax^4+4bx^3+6cx^2+4dx+e=0$ may have two pairs of equal roots.

Solution : Let $\alpha,\alpha,\beta,\beta$ be the roots of $ax^4+4bx^3+6cx^2+4dx+e=0$

From the relations of the coefficients and the roots, we have,

$$\alpha+\alpha+\beta+\beta = -\frac{4b}{a}$$

$$\alpha\alpha+\alpha\beta+\alpha\beta+\alpha\beta+\alpha\beta+\beta\beta = \frac{6c}{a}$$

$$\alpha\alpha\beta+\alpha\alpha\beta+\alpha\beta\beta+\alpha\beta\beta = -\frac{4d}{a}$$

$$\alpha\alpha\beta\beta = \frac{e}{a}$$

The above equations can be rewritten as

$$2(\alpha+\beta) = -\frac{4b}{a} \Rightarrow \alpha+\beta = -\frac{2b}{a} \quad \text{-----} \quad (6.41),$$

$$\alpha^2+\beta^2+4\alpha\beta = \frac{6c}{a} \quad \text{-----} \quad (6.42)$$

$$2(\alpha^2\beta+\alpha\beta^2) = -\frac{4d}{a} \quad \text{-----} \quad (6.43)$$

$$\alpha^2\beta^2 = \frac{e}{a} \quad \text{-----} \quad (6.44)$$

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Now from (6.41), $\alpha+\beta = -\dfrac{2b}{a}$,

and from (6.43), $2(\alpha^2\beta+\alpha\beta^2) = -\dfrac{4d}{a}$

(i.e) $2\alpha\beta(\alpha+\beta) = -\dfrac{4d}{a}$

(i.e) $2\alpha\beta\left(-\dfrac{2b}{a}\right) = -\dfrac{4d}{a}$

(i.e) $\alpha\beta = \dfrac{d}{b}$ -- (6.45)

Now α, β are the roots of the equation $x^2-(\alpha+\beta)x+\alpha\beta=0$

(i.e) $x^2-\left(-\dfrac{2b}{a}\right)x+\left(\dfrac{d}{b}\right)=0$

(i.e) $x^2+\left(\dfrac{2b}{a}\right)x+\left(\dfrac{d}{b}\right)=0$

$$\therefore ax^4+4bx^3+6cx^2+4dx+e = a\left[x^2+\left(\frac{2b}{a}\right)x+\left(\frac{d}{b}\right)\right]^2$$

$$= a\left[x^4+\left(\frac{4b^2}{a^2}\right)x^2+\frac{d^2}{b^2}+2x^2\left(\frac{2b}{a}\right)+2x^2\left(\frac{d}{b}\right)+2\left(\frac{2bx}{a}\right)\left(\frac{d}{b}\right)\right]$$

$$= a\left[x^4+\left(\frac{4b}{a}\right)x^3+\left(\frac{4b^2}{a^2}+\frac{2d}{b}\right)x^2+\frac{4d}{a}x+\frac{d^2}{b^2}\right]$$

Comparing like powers of x on both sides, we get,

$6c = a\left(\dfrac{4b^2}{a^2}+\dfrac{2d}{b}\right)$ and $e=\dfrac{ad^2}{b^2}$

(i.e) $6c=2a\left(\dfrac{2b^2+a^2d}{a^2b}\right)$ and $e=\dfrac{ad^2}{b^2}$

(i.e) $3abc=3b^3+a^2d$ and $b^2e=ad^2$

which is the required condition.

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Example 6.3.5 :

Solve the equation $4x^3+20x^2-23x+6=0$, two of its roots being equal.

Solution : Let α,α,β be the roots of $4x^3+20x^2-23x+6=0$

From the relations of the coefficients and the roots, we have,

$$\alpha+\alpha+\beta = -\frac{20}{4} = -5 \text{ ------------------- } (6.46)$$

$$\alpha\alpha+\alpha\beta+\alpha\beta = -\frac{23}{4} \Rightarrow \alpha^2+2\alpha\beta = -\frac{23}{4} \text{ --------- } (6.47)$$

$$\alpha\alpha\beta = \frac{6}{4} \Rightarrow \alpha^2\beta = \frac{3}{2} \text{ ------------------- } (6.48)$$

From (6.46), we get, $\beta=-5-2\alpha$ ------------------------------- (6.49)

From (6.47) and (6.49), we have, $\alpha^2+2\alpha(-5-2\alpha) = -\frac{23}{4}$

(i.e) $\alpha^2-10\alpha-4\alpha^2 = -\frac{23}{4}$

(i.e) $-10\alpha-3\alpha^2 = -\frac{23}{4}$

(i.e) $-40\alpha-12\alpha^2=-23$

(i.e) $12\alpha^2+40\alpha-23=0$

(i.e) $(6\alpha+23)(2\alpha-1)=0$

(i.e) $6\alpha+23=0$ or $2\alpha-1=0$

(i.e) $\alpha=-\frac{23}{6}$ or $\alpha=\frac{1}{2}$

Using (6.49), and putting $\alpha=-\frac{23}{6}$, we get, $\beta=-5-2\left(-\frac{23}{6}\right)$

(i.e) $\beta=\frac{8}{3}$

Now $\alpha=-\frac{23}{6}$ and $\beta=\frac{8}{3}$ is not satisfies the equation (6.48)

Thus consider $\alpha=\frac{1}{2}$ and putting in (6.48), we get, $\beta=-5-2\left(\frac{1}{2}\right)$

(i.e) $\beta=-6$

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Now $\alpha = \frac{1}{2}$ and $\beta = -6$ is satisfies the equation (6.48)

Thus the required roots are $\frac{1}{2}, \frac{1}{2}, -6$.

Example 6.3.5 :

Solve the equation $x^4 + 4x^3 - 2x^2 - 12x + 9 = 0$ which has two pairs.

Solution : Let $\alpha, \alpha, \beta, \beta$ be the roots of $x^4 + 4x^3 - 2x^2 - 12x + 9 = 0$

From the relations of the coefficients and the roots, we have,

$\alpha + \alpha + \beta + \beta = -4 \Rightarrow 2(\alpha + \beta) = -4 \Rightarrow \alpha + \beta = -2$ ------------- (6.50)

$\alpha\alpha + \alpha\beta + \alpha\beta + \alpha\beta + \alpha\beta + \beta\beta = -2 \Rightarrow \alpha^2 + \beta^2 + 4\alpha\beta = -2$ ---- (6.51)

$\alpha\alpha\beta + \alpha\alpha\beta + \alpha\beta\beta + \alpha\beta\beta = 12 \Rightarrow 2(\alpha^2\beta + \alpha\beta^2) = 12$ ---------- (6.52)

$\alpha\alpha\beta\beta = 9 \Rightarrow \alpha^2\beta^2 = 9$ -- (6.53)

Now from (6.51), $\alpha^2 + \beta^2 + 4\alpha\beta = -2$

(i.e) $(\alpha + \beta)^2 + 2\alpha\beta = -2$

(i.e) $(-2)^2 + 2\alpha\beta = -2$ (using (6.50))

(i.e) $4 + 2\alpha\beta = -2$

(i.e) $2\alpha\beta = -6$

(i.e) $\alpha\beta = -3$ --- (6.54)

From (6.50) and (6.54), we have, $\alpha(-2-\alpha) = -3$

(i.e) $\alpha^2 + 2\alpha - 3 = 0$

(i.e) $(\alpha - 1)(\alpha + 3) = 0$

(i.e) $\alpha = 1$ or $\alpha = -3$

When $\alpha = 1$, then from (6.54), we have, $\beta = -3$

When $\alpha = -3$, then from (6.54), we have, $\beta = 1$

Thus the roots of the equation are $-3, -3, 1, 1$

Example 6.3.6 :

Solve the equation $x^3 - 9x^2 + 14x + 24 = 0$ two of its roots being in the ratio $3:2$.

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Solution : Let $3\alpha, 2\alpha, \beta$ be the roots of $x^3 - 9x^2 + 14x + 24 = 0$

From the relations of the coefficients and the roots, we have,

$$3\alpha + 2\alpha + \beta = 9 \Rightarrow 5\alpha + \beta = 9 \Rightarrow \beta = 9 - 5\alpha \text{ ---------------------- } (6.55)$$

$$3\alpha 2\alpha + 3\alpha\beta + 2\alpha\beta = 14 \Rightarrow 6\alpha^2 + 5\alpha\beta = 14 \text{ ------------------------ } (6.56)$$

$$3\alpha 2\alpha\beta = -24 \Rightarrow 6\alpha^2\beta = -24 \text{ --------------------------------------- } (6.57)$$

Now from (6.55) and (6.56), we have, $6\alpha^2 + 5\alpha(9 - 5\alpha) = 14$

(i.e) $6\alpha^2 + 45\alpha - 25\alpha^2 = 14$

(i.e) $19\alpha^2 - 45\alpha + 14 = 0$

(i.e) $\alpha = 2$ or $\alpha = \dfrac{7}{19}$

When $\alpha = 2$ then (6.55) implies $\beta = -1$

When $\alpha = \dfrac{7}{19}$ then (6.55) implies $\beta = \dfrac{136}{19}$ which not satisfies (6.57)

Thus the required roots are 6, 4, – 1

Example 6.3.7 :

Solve the equation $x^4 + 2x^3 - 21x^2 - 22x + 40 = 0$ whose roots are in arithmetic progression.

Solution : Let $a - 3d, a - d, a + d, a + 3d$ be the roots of $x^4 + 2x^3 - 21x^2 - 22x + 40 = 0$.

From the relations of the coefficients and the roots, we have,

$$a - 3d + a - d + a + da + 3d = -2 \Rightarrow 4a = -2 \Rightarrow a = -\frac{1}{2} \text{ ------- } (6.58)$$

$$(a - 3d)(a - d)(a + d)(a + 3d) = 40 \Rightarrow (a^2 - 9d^2)(a^2 - d^2) = 40$$

(i.e) $\left(\dfrac{1}{4} - 9d^2\right)\left(\dfrac{1}{4} - d^2\right) = 40$

(i.e) $\left(\dfrac{1}{4} - 9z\right)\left(\dfrac{1}{4} - z\right) = 40$ where $z = d^2$

(i.e) $(1 - 36z)(1 - 4z) = 640$

(i.e) $1 - 4z - 36z + 144z^2 = 640$

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(i.e) $144z^2 - 40z - 639 = 0$

(i.e) $(4z-9)(36z+71) = 0$

(i.e) $z = \frac{9}{4}$ or $z = -\frac{71}{36}$

Now $z = \frac{9}{4} \Rightarrow d^2 = \frac{9}{4} \Rightarrow d = \pm\frac{3}{2}$

Thus when $a = -\frac{1}{2}$ and $d = \frac{3}{2}$, then the roots are $-5, -2, 1, 4$.

and when $a = -\frac{1}{2}$ and $d = -\frac{3}{2}$, then the roots are $4, 1, -2, 5$.

Example 6.3.8 :

Solve the equation $6x^3 - 11x^2 - 3x + 2 = 0$ whose roots are in Harmonic progression.

Solution : Let α, β, γ be the roots of $6x^3 - 11x^2 - 3x + 2 = 0$.

Since the roots are in harmonic progression, then $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are in arithmetic progression.

(i.e) $\frac{2}{\beta} = \frac{1}{\alpha} + \frac{1}{\gamma}$

(i.e) $2\alpha\gamma = \alpha\beta + \beta\gamma$ -- (6.59)

From the relations of the coefficients and the roots, we have,

$\alpha + \beta + \gamma = \frac{11}{6}$ -- (6.60)

$\alpha\beta + \beta\gamma + \gamma\alpha = -\frac{3}{6} \Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = -\frac{1}{2}$ ----------------- (6.61)

$\alpha\beta\gamma = -\frac{2}{6} \Rightarrow \alpha\beta\gamma = -\frac{1}{3}$ --- (6.62)

From (6.59) and (6.61), we have, $2\alpha\gamma + \alpha\gamma = -\frac{1}{2}$

(i.e) $3\alpha\gamma = -\frac{1}{2}$

(i.e) $\alpha\gamma = -\frac{1}{6}$ -- (6.63)

From (6.62) and (6.63), we have, $-\frac{1}{6}\beta=-\frac{1}{3}$

(i.e) $\beta=2$ -- (6.64)

From (6.60) and (6.64), we get, $\alpha+2+\gamma=\frac{11}{6}$

(i.e) $\gamma=-\frac{1}{6}-\alpha$ -- (6.65)

From (6.63) and (6.65), we have, $\alpha\left(-\frac{1}{6}-\alpha\right)=-\frac{1}{6}$

(i.e) $\alpha\left(\frac{-1-6\alpha}{6}\right)=-\frac{1}{6}$

(i.e) $-\alpha-6\alpha^2=-1$

(i.e) $6\alpha^2+\alpha-1=0$

(i.e) $(2\alpha+1)(3\alpha-1)=0$

(i.e) $\alpha=-\frac{1}{2}$ or (i.e) $\alpha=\frac{1}{3}$

When $\alpha=-\frac{1}{2}$ then from (6.65), $\gamma=-\frac{1}{6}+\frac{1}{2}=\frac{1}{3}$

When $\alpha=\frac{1}{3}$ then from (6.65), $\gamma=-\frac{1}{6}-\frac{1}{3}=-\frac{1}{2}$

Thus $\alpha=-\frac{1}{2}$ and $\gamma=\frac{1}{3}$, then the roots of the equation are $-\frac{1}{2},2,\frac{1}{3}$.

Example 6.3.9 :

Given that two of the roots of $45x^4-54x^3-98x^2+150x-75=0$ are equal in absolute value but opposite in sign. Solve the equation completely.

Solution : Let α, β, γ, δ be the roots of $45x^4-54x^3-98x^2+150x-75=0$ such that $\alpha=-\beta$

(i.e) $\alpha+\beta=0$ -- (6.66)

Now sum of the roots = $\alpha+\beta+\gamma+\delta=\frac{54}{45}$

(i.e) $\gamma+\delta=\frac{54}{45}=\frac{6}{5}$ --- (6.67)

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Since $\alpha, \beta, \gamma, \delta$ be the roots of the equation, then, we have,

$$x^4 - \frac{54}{45}x^3 - \frac{98}{45}x^2 + \frac{150}{45}x - \frac{75}{45} = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$$

$$= \left[x^2 - (\alpha \pm \beta)x + \alpha\beta\right]\left[x^2 - (\gamma+\delta)x + \gamma\delta\right]$$

$$= \left[x^2 + \lambda\right]\left[x^2 - \frac{6}{5}x + \mu\right] \text{ where } \lambda = \alpha\beta \text{ and } \mu = \gamma\delta \text{ ---- (6.68)}$$

$$= x^4 - \frac{6}{5}x^3 + \mu x^2 + \lambda x^2 - \frac{6}{5}\lambda x + \lambda\mu$$

$$= x^4 - \frac{6}{5}x^3 + (\mu+\lambda)x^2 - \frac{6}{5}\lambda x + \lambda\mu$$

Comparing like powers of x on both sides, we get,

$$\lambda + \mu = -\frac{98}{45}, \quad -\frac{6}{5}\lambda = \frac{150}{45} \text{ and } \lambda\mu = -\frac{75}{45} \text{ ---------------------------- (6.69)}$$

Now $-\frac{6}{5}\lambda = \frac{150}{45} \Rightarrow \lambda = -\frac{25}{9}$ -- (6.70)

From (6.69) and (6.70), we have, $\mu = \frac{3}{5}$

Thus (6.68) becomes $\left[x^2 + \lambda\right]\left[x^2 - \frac{6}{5}x + \mu\right] = 0$

(i.e) $\left[x^2 - \frac{25}{9}\right]\left[x^2 - \frac{6}{5}x + \frac{3}{5}\right] = 0$

(i.e) $\left[x^2 - \frac{25}{9}\right] = 0$ or $\left[x^2 - \frac{6}{5}x + \frac{3}{5}\right] = 0$

Now $\left[x^2 - \frac{25}{9}\right] = 0 \Rightarrow x = \pm\frac{5}{3}$

and $\left[x^2 - \frac{6}{5}x + \frac{3}{5}\right] = 0$

$\Rightarrow 5x^2 - 6x + 3 = 0$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{6 \pm \sqrt{36-60}}{10}$$

$$\Rightarrow x = \frac{6 \pm \sqrt{-24}}{10}$$

$$\Rightarrow x = \frac{6 \pm i2\sqrt{6}}{10}$$

$$\Rightarrow x = \frac{3 \pm i\sqrt{6}}{5}$$

Thus the roots are $\pm\frac{5}{3}, \frac{3 \pm i\sqrt{6}}{5}$

Check your progress

Questions :

(1) Solve the equation $x^3 - 3x^2 + 4 = 0$, two of its roots being equal.

(2) Solve the equation $81x^3 - 18x^2 - 36x + 8 = 0$ whose roots are in harmonic progression.

(answer (2) : $\frac{2}{9}, \frac{2}{3}, -\frac{2}{3}$)

(3) If two roots of the equation x^4 [illegible] $+ qx$ [illegible] $= 0$ are equal in value but differ in sign, show that $r^2 + p$ [illegible].

(4) Solve $x^4 - 8x^3 + 14x^2 + 8x - 15 = 0$, it being given that the sum of two of the roots is equal to the sum of the other two.

6.4. Symmetric function of the roots

If a function involving all the roots of an equation is unaltered in value if any two of the roots are interchanged, it is called a symmetric function of the roots.

Let $\alpha_1, \alpha_2, \alpha_3, \mathrm{L}\ , \alpha_n$ be the roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \mathrm{L}\ + p_n$$

Then we know that

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$S_1 = \sum \alpha_1 = -p_1$

$S_2 = \sum \alpha_1\alpha_2 = p_2$

$S_3 = \sum \alpha_1\alpha_2\alpha_3 = -p_3$ and so on.

Without working the values of the roots separately in terms of the coefficients, by using the relations between the coefficients and the roots of an equation, we can express any symmetric function of the roots in terms of the coefficients of the equations.

Example 6.4.1 :

If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$ find the value of

(i) $\alpha^2 + \beta^2 + \gamma^2$

(ii) $\alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma + \gamma^2\alpha + \gamma^2\beta$

(iii) $\alpha^3 + \beta^3 + \gamma^3$

(iv) $\sum \alpha^2\beta\gamma$

(v) $\sum \alpha^2\beta^2$

(vi) $\sum \alpha^3\beta$

(vii) $\sum \alpha^4$

(viii) $\sum \frac{1}{\alpha^2\beta^2}$

(ix) $\frac{\beta^2+\gamma^2}{\beta+\gamma} + \frac{\gamma^2+\alpha^2}{\gamma+\alpha} + \frac{\alpha^2+\beta^2}{\alpha+\beta}$

(x) $(\beta+\gamma-\alpha)^3 + (\gamma+\alpha-\beta)^3 + (\alpha+\beta-\gamma)^3$

(xi) $\frac{\alpha\beta}{\gamma} + \frac{\beta\gamma}{\alpha} + \frac{\gamma\alpha}{\beta}$

Solution : Given that α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$

$\therefore\ \alpha + \beta + \gamma = \sum \alpha = -p$ -- (6.71)

$\alpha\beta + \beta\gamma + \gamma\alpha = \sum \alpha\beta = q$ ---------------------------------- (6.72)

$\alpha\beta\gamma = -r$ -- (6.73)

(i) $\alpha^2+\beta^2+\gamma^2 = \sum\alpha^2$

$$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$$

$$= (-p)^2 - 2(q)$$

$$= p^2 - 2q$$

(ii) $\alpha^2\beta+\alpha^2\gamma+\beta^2\alpha+\beta^2\gamma+\gamma^2\alpha+\gamma^2\beta$

$$= \sum\alpha^2\beta$$

$$= \left(\sum\alpha\right)\left(\sum\alpha\beta\right) - 3\alpha\beta\gamma$$

$$= (-p)(q) - 3(-r)$$

$$= 3r - pq$$

(iii) $\alpha^3+\beta^3+\gamma^3 = \sum\alpha^3$

$$= \left(\sum\alpha\right)\left(\sum\alpha^2\right) - \left(\sum\alpha^2\beta\right)$$

$$= (-p)(p^2-2q) - (3r-pq)$$

$$= -p^3 + 2pq - 3r + pq$$

$$= 3pq - p^3 - 3r$$

(iv) $\sum\alpha^2\beta\gamma = \alpha^2\beta\gamma+\beta^2\gamma\alpha+\gamma^2\alpha\beta$

$$= \alpha\beta\gamma(\alpha+\beta+\gamma)$$

$$= (-r)(-p)$$

$$= pr$$

(v) $\sum\alpha^2\beta^2 = \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2$

$$= \left(\sum\alpha\beta\right)^2 - 2\sum\alpha^2\beta\gamma$$

$$= q^2 - pr$$

(vi) $\sum\alpha^3\beta = \alpha^3\beta+\alpha^3\gamma+\beta^3\alpha+\beta^3\gamma+\gamma^3\alpha+\gamma^3\beta$

$$= \left(\sum\alpha^2\right)\left(\sum\alpha\beta\right) - \left(\sum\alpha^2\beta\gamma\right)$$

$$= q(p^2-2q) - pr$$

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$$= p^2q - 2q^2 - pr$$

(vii) $\sum\alpha^4 = \alpha^4 + \beta^4 + \gamma^4$

$$= \left(\sum\alpha^2\right)^2 - 2\left(\sum\alpha^2\beta^2\right)$$

$$= (p^2 - 2q)^2 - 2(q^2 - 2pr)$$

$$= p^4 - 4p^2q + 4q^2 - 2q^2 + 4pr$$

$$= p^4 - 4p^2q + 2q^2 + 4pr$$

(viii) $\sum\dfrac{1}{\alpha^2\beta^2} = \dfrac{1}{\alpha^2\beta^2} + \dfrac{1}{\beta^2\gamma^2} + \dfrac{1}{\gamma^2\alpha^2}$

$$= \frac{\gamma^2 + \alpha^2 + \beta^2}{\alpha^2\beta^2\gamma^2}$$

$$= \frac{\sum\alpha^2}{(\alpha\beta\gamma)^2}$$

$$= \frac{p^2 - 2q}{r^2}$$

(ix) $\sum\dfrac{\beta^2 + \gamma^2}{\beta + \gamma} = \dfrac{\beta^2 + \gamma^2}{\beta + \gamma} + \dfrac{\gamma^2 + \alpha^2}{\gamma + \alpha} + \dfrac{\alpha^2 + \beta^2}{\alpha + \beta}$

$$= \sum\left(\frac{(\alpha + \beta)(\alpha + \gamma)(\beta^2 + \gamma^2)}{(\alpha + \beta)(\alpha + \gamma)(\beta + \gamma)}\right)$$

$$= \frac{\left(\sum(\alpha + \beta)(\alpha + \gamma)(\beta^2 + \gamma^2)\right)}{(\alpha + \beta)(\alpha + \gamma)(\beta + \gamma)}$$

Now $\sum(\alpha + \beta)(\alpha + \gamma)(\beta^2 + \gamma^2)$

$$= \frac{\sum(\alpha + \beta)(\alpha + \gamma)(\beta^2 + \gamma^2)}{\sum(}$$

(x) $\sum(\beta + \gamma - \alpha)^3 = (\beta + \gamma - \alpha)^3 + (\gamma + \alpha - \beta)^3 + (\alpha + \beta - \gamma)^3$

Let $A = \alpha + \beta - \gamma$, $B = \beta + \gamma - \alpha$, $C = \gamma + \alpha - \beta$

Now $B + C = \beta + \gamma - \alpha + \gamma + \alpha - \beta = 2\gamma$

Similarly $C + A = 2\alpha$ and $A + B = 2\beta$

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Again $A+B+C = \alpha+\beta-\gamma+\beta+\gamma-\alpha+\gamma+\alpha-\beta$

$$= \alpha+\beta+\gamma$$

Thus $\sum(\beta+\gamma-\alpha)^3 = (\beta+\gamma-\alpha)^3+(\gamma+\alpha-\beta)^3+(\alpha+\beta-\gamma)^3$

$$= A^3+B^3+C^3$$

$$= (A+B+C)^3-3(A+B)(B+C)(C+A)$$

$$= \left(\sum\alpha\right)^3-3(2\beta)(2\gamma)(2\alpha)$$

$$= (-p)^3-24(-r)$$

$$= 24r-p^3$$

(xi) $\dfrac{\alpha\beta}{\gamma}+\dfrac{\beta\gamma}{\alpha}+\dfrac{\gamma\alpha}{\beta} = \dfrac{(\alpha\beta)^2+(\beta\gamma)^2+(\alpha\gamma)^2}{\alpha\beta\gamma}$

$$= \frac{\sum(\alpha\beta)^2}{\alpha\beta\gamma}$$

$$= \frac{q^2-2pr}{-r}$$

$$= \frac{2pr-q^2}{r}$$

Example 6.4.2 :

If α,β,γ are the roots of $x^3+px^2+qx+r=0$ find the value of $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$

Solution : Given that α, β, γ are the roots of $x^3+px^2+qx+r=0$

$\therefore\ \alpha+\beta+\gamma = \sum\alpha = -p$

$\alpha\beta+\beta\gamma+\gamma\alpha = \sum\alpha\beta = q$

$\alpha\beta\gamma = -r$

$(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$

$$= (\alpha+\beta+\gamma-\gamma)(\alpha+\beta+\gamma-\alpha)(\gamma+\alpha+\beta-\beta)$$

$$= (-p-\gamma)(-p-\alpha)(-p-\beta)$$

$$= (-1)(p+\gamma)(p+\alpha)(p+\beta)$$

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$$= (-1)(p+\gamma)(p^2 + p\beta + p\alpha + \alpha\beta)$$

$$= (-1)(p^3 + p^2\beta + p^2\alpha + p\alpha\beta + p^2\gamma + p\beta\gamma + p\alpha\gamma + \alpha\beta\gamma)$$

$$= (-1)(p^3 + p^2(\alpha+\beta+\gamma) + p(\alpha\beta+\beta\gamma+\gamma\alpha) + \alpha\beta\gamma)$$

$$= (-1)(p^3 + p^2(-p) + pq + (-r))$$

$$= r - pq$$

Thus $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha) = r - pq$

This proves the problem.

Example 6.4.3 :

If α, β, γ are the roots of $x^3 + qx + r = 0$ find the value of

(i) $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$

(ii) $\dfrac{1}{\beta+\gamma} + \dfrac{1}{\gamma+\alpha} + \dfrac{1}{\alpha+\beta}$

(iii) $\left(\dfrac{1}{\beta} + \dfrac{1}{\gamma} - \dfrac{1}{\alpha}\right)\left(\dfrac{1}{\gamma} + \dfrac{1}{\alpha} - \dfrac{1}{\beta}\right)\left(\dfrac{1}{\alpha} + \dfrac{1}{\beta} - \dfrac{1}{\gamma}\right)$

Solution : Given that α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$

$\therefore\ \alpha+\beta+\gamma = \sum\alpha = 0$

$\alpha\beta+\beta\gamma+\gamma\alpha = \sum\alpha\beta = q$

$\alpha\beta\gamma = -r$

(i) $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$

$$= (\alpha+\beta+\gamma-\gamma)(\alpha+\beta+\gamma-\alpha)(\gamma+\alpha+\beta-\beta)$$

$$= (0-\gamma)(0-\alpha)(0-\beta)$$

$$= -\alpha\beta\gamma$$

$$= -(-r)$$

$$= r$$

Thus $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha) = r$

(ii) $\dfrac{1}{\beta+\gamma} + \dfrac{1}{\gamma+\alpha} + \dfrac{1}{\alpha+\beta}$

$$= \frac{1}{\alpha+\beta+\gamma-\alpha} + \frac{1}{\beta+\gamma+\alpha-\beta} + \frac{1}{\gamma+\alpha+\beta-\gamma}$$

$$= \frac{1}{0-\alpha} + \frac{1}{0-\beta} + \frac{1}{0-\gamma}$$

$$= (-1)\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)$$

$$= (-1)\left(\frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}\right)$$

$$= (-1)\left(\frac{q}{-r}\right)$$

$$= \frac{q}{r}$$

Hence $\frac{1}{\beta+\gamma} + \frac{1}{\gamma+\alpha} + \frac{1}{\alpha+\beta} = \frac{q}{r}$

and (iii) $\left(\frac{1}{\beta} + \frac{1}{\gamma} - \frac{1}{\alpha}\right)\left(\frac{1}{\gamma} + \frac{1}{\alpha} - \frac{1}{\beta}\right)\left(\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\gamma}\right)$

$$= \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} - 2\frac{1}{\alpha}\right)\left(\frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\alpha} - 2\frac{1}{\beta}\right)\left(\frac{1}{\gamma} + \frac{1}{\alpha} + \frac{1}{\beta} - 2\frac{1}{\gamma}\right)$$

Now $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = \frac{q}{-r} = -\frac{q}{r}$

Thus $\left(\frac{1}{\beta} + \frac{1}{\gamma} - \frac{1}{\alpha}\right)\left(\frac{1}{\gamma} + \frac{1}{\alpha} - \frac{1}{\beta}\right)\left(\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\gamma}\right)$

$$= \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} - 2\frac{1}{\alpha}\right)\left(\frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\alpha} - 2\frac{1}{\beta}\right)\left(\frac{1}{\gamma} + \frac{1}{\alpha} + \frac{1}{\beta} - 2\frac{1}{\gamma}\right)$$

$$= \left(-\frac{q}{r} - 2\frac{1}{\alpha}\right)\left(-\frac{q}{r} - 2\frac{1}{\beta}\right)\left(-\frac{q}{r} - 2\frac{1}{\gamma}\right)$$

$$= (-1)\left(\frac{q}{r} + 2\frac{1}{\alpha}\right)\left(\frac{q}{r} + 2\frac{1}{\beta}\right)\left(\frac{q}{r} + 2\frac{1}{\gamma}\right)$$

$$= (-1)\left(\frac{q}{r} + 2\frac{1}{\alpha}\right)\left(\frac{q^2}{r^2} + 2\frac{q}{r}\frac{1}{\gamma} + 2\frac{q}{r}\frac{1}{\beta} + 4\frac{1}{\beta}\frac{1}{\gamma}\right)$$

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$$=(-1)\left(\frac{q^3}{r^3}+\frac{2q^2}{r^2}\frac{1}{\gamma}+\frac{2q^2}{r^2}\frac{1}{\beta}+\frac{4q}{r}\frac{1}{\beta\gamma}+\frac{2q^2}{r^2}\frac{1}{\alpha}+\frac{4q}{r}\frac{1}{\alpha\gamma}+\frac{4q}{r}\frac{1}{\alpha\beta}+8\frac{1}{\alpha\beta\gamma}\right)$$

$$=(-1)\left(\frac{q^3}{r^3}+\frac{2q^2}{r^2}\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)+\frac{4q}{r}\left(\frac{1}{\beta\gamma}+\frac{1}{\alpha\gamma}+\frac{1}{\alpha\beta}\right)+8\frac{1}{\alpha\beta\gamma}\right)$$

$$=(-1)\left(\frac{q^3}{r^3}+\frac{2q^2}{r^2}\left(-\frac{q}{r}\right)+\frac{4q}{r}\left(\frac{\alpha}{\alpha\beta\gamma}+\frac{\beta}{\alpha\beta\gamma}+\frac{\gamma}{\alpha\beta\gamma}\right)+8\left(\frac{1}{-r}\right)\right)$$

$$=(-1)\left(\frac{q^3}{r^3}-\frac{2q^3}{r^3}+\frac{4q}{r}(0)-\frac{8}{r}\right)$$

$$=(-1)\left(-\frac{q^3}{r^3}-\frac{8}{r}\right)$$

$$=\frac{q^3}{r^3}+\frac{8}{r}$$

$$=\frac{q^3+8r^2}{r^3}$$

Hence $\left(\frac{1}{\beta}+\frac{1}{\gamma}-\frac{1}{\alpha}\right)\left(\frac{1}{\gamma}+\frac{1}{\alpha}-\frac{1}{\beta}\right)\left(\frac{1}{\alpha}+\frac{1}{\beta}-\frac{1}{\gamma}\right)=\frac{q^3+8r^2}{r^3}$

Example 6.4.4 :

Find the sum of the cubes of the roots of $x^4-22x^2+84x-49=0$.

Solution : Given that $\alpha, \beta, \gamma, \delta$ are the roots of $x^4-22x^2+84x-49=0$

$\therefore\ \sum\alpha = 0$

$\sum\alpha\beta = -22$

$\sum\alpha\beta\gamma = -84$

$\alpha\beta\gamma\delta = -49$

Now $\left(\sum\alpha\right)\left(\sum\alpha^2\right)=\sum\alpha^3+\sum\alpha^2\beta$ ------------------- (6.74)

and $\left(\sum\alpha\right)\left(\sum\alpha\beta\right)=\sum\alpha^2\beta+3\sum\alpha\beta\gamma$

$\Rightarrow \sum\alpha^2\beta=\left(\sum\alpha\right)\left(\sum\alpha\beta\right)-3\sum\alpha\beta\gamma$

$$\Rightarrow \sum \alpha^2\beta = 0-3(-84)$$

$$\Rightarrow \sum \alpha^2\beta = 252$$

$$\therefore (6.74) \Rightarrow 0 = \sum \alpha^3 + 252$$

$$\Rightarrow \sum \alpha^3 = -252$$

Example 6.4.5 :

Show that for the cubic equation $a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$,

$$(\beta-\gamma)^2 + (\gamma-\alpha)^2 + (\alpha-\beta)^2 = \frac{18}{a_0^2}\left(a_1^2 - a_0a_2\right).$$

Proof : Given that α, β, γ are the roots of $a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$

$$\therefore \sum \alpha = -\frac{3a_1}{a_0}$$

$$\sum \alpha\beta = \frac{3a_2}{a_0}$$

$$\alpha\beta\gamma = -\frac{a_3}{a_0}$$

Now $(\beta-\gamma)^2 + (\gamma-\alpha)^2 + (\alpha-\beta)^2$

$$= \beta^2 + \gamma^2 - 2\beta\gamma + \gamma^2 + \alpha^2 - 2\gamma\alpha + \alpha^2 + \beta^2 - 2\alpha\beta$$

$$= 2\sum \alpha^2 - 2\sum \alpha\beta$$

$$= 2\left[\left(\sum \alpha\right)^2 - 2\sum \alpha\beta\right] - 2\sum \alpha\beta$$

$$= 2\left(\sum \alpha\right)^2 - 6\sum \alpha\beta$$

$$= 2\frac{9a_1^2}{a_0^2} - 6\left(\frac{3a_2}{a_0}\right)$$

$$= \frac{18}{a_0^2}\left(a_1^2 - a_0a_2\right)$$

This proves the problem.

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Summary

In this unit we have learned that the method of finding imaginary roots, rational roots, relation between roots and coefficients and symmetric functions of the roots

Further Reading

You can also refer the following books for further reading.

(1) Algebra by T.K.Manicavachagom Pillai and others Vol I

(2) Classical Algebra by Arumugam and others.

UNIT VII

RELATIONS BETWEEN ROOTS AND COEFFICIENTS

Space for Hints

Unit Objectives

Unit Structure

Check your progress

Summary

Further Reading

Space for Hints

Objectives :

In this unit, we are going to discuss to find the sum of the power of the roots of an equation, Newton's theorem, transformations of equations, roots multiplied by a given number, solving reciprocal equations, and finding equations by increasing or decreasing roots by a given quantity.

After completing this unit, students may able to know

- o Sum of the powers of the roots of an equation
- o Newton's theorem
- o Transformation of equations
- o Solving reciprocal equations
- o Roots multiplied by a number

7.1. Sum of the powers of the roots of an equation

We shall find the formula for finding the sum of r^{th} power of roots of an equation.

Let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ be the roots of an equation $f(x)=0$.

Let S_r be denote the sum of n^{th} roots of the equation $f(x)=0$

(i.e) $S_r = \alpha_1^r + \alpha_2^r + \alpha_3^r + \ldots + \alpha_n^r$

We use the following easy method to find S_r when $r > 4$.

Now $f(x) = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)$

Taking logarithms on both sides, we get,

$\log f(x) = \log(x-\alpha_1)+\log(x-\alpha_2)+\log(x-\alpha_3)+\ldots+\log(x-\alpha_n)$

Differentiating the above equation with respect to x, we get,

$$\frac{f'(x)}{f(x)} = \frac{1}{x-\alpha_1}+\frac{1}{x-\alpha_2}+\frac{1}{x-\alpha_3}+\ldots+\frac{1}{x-\alpha_n}$$

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$$x\frac{f'(x)}{f(x)} = \frac{x}{x-\alpha_1}+\frac{x}{x-\alpha_2}+\frac{x}{x-\alpha_3}+\cdots+\frac{x}{x-\alpha_n}$$

$$= \frac{1}{1-\frac{\alpha_1}{x}}+\frac{1}{1-\frac{\alpha_2}{x}}+\frac{1}{1-\frac{\alpha_3}{x}}+\cdots+\frac{1}{1-\frac{\alpha_n}{x}}$$

$$= \left(1-\frac{\alpha_1}{x}\right)^{-1}+\left(1-\frac{\alpha_2}{x}\right)^{-1}+\left(1-\frac{\alpha_3}{x}\right)^{-1}+\cdots+\left(1-\frac{\alpha_n}{x}\right)^{-1}$$

$$= 1+\frac{\alpha_1}{x}+\frac{\alpha_1^2}{x^2}+\frac{\alpha_1^3}{x^3}+\cdots+\frac{\alpha_1^n}{x^n}+\cdots$$

$$+\ 1+\frac{\alpha_2}{x}+\frac{\alpha_2^2}{x^2}+\frac{\alpha_2^3}{x^3}+\cdots+\frac{\alpha_2^n}{x^n}+\cdots$$

$$+\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$+\ 1+\frac{\alpha_n}{x}+\frac{\alpha_n^2}{x^2}+\frac{\alpha_n^3}{x^3}+\cdots+\frac{\alpha_n^n}{x^n}+\cdots$$

$$= n+\left(\sum\alpha_1\right)\frac{1}{x}+\left(\sum\alpha_1^2\right)\frac{1}{x^2}+\cdots+\left(\sum\alpha_1^r\right)\frac{1}{x^r}+\cdots$$

$$= n+S_1\frac{1}{x}+S_2\frac{1}{x^2}+\cdots+S_r\frac{1}{x^r}+\cdots$$

Thus S_r = coefficient of $\frac{1}{x^r}$ in the expansion of $x\frac{f'(x)}{f(x)}$.

7.2. Newton's theorem on the sum of the powers of the roots

Let $\alpha_1,\alpha_2,\alpha_3,\cdots,\alpha_n$ be the roots of an equation

$f(x)=x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_n=0$ ------------------------(7.1).

Let S_r be denote the sum of n^{th} roots of the equation $f(x)=0$

(i.e) $S_r = \alpha_1^r+\alpha_2^r+\alpha_3^r+\cdots+\alpha_n^r$ and $S_0=n$

Now $f(x) = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n)$

Space for Hints

Taking logarithms on both sides, we get,

$$\log f(x) = \log(x-\alpha_1)+\log(x-\alpha_2)+\log(x-\alpha_3)+\cdots+\log(x-\alpha_n)$$

Differentiating the above equation with respect to x, we get,

$$\frac{f'(x)}{f(x)} = \frac{1}{x-\alpha_1}+\frac{1}{x-\alpha_2}+\frac{1}{x-\alpha_3}+\cdots+\frac{1}{x-\alpha_n}$$

$$f'(x) = \frac{f(x)}{x-\alpha_1}+\frac{f(x)}{x-\alpha_2}+\frac{f(x)}{x-\alpha_3}+\cdots+\frac{f(x)}{x-\alpha_n}$$

By actual division, we obtain,

$$\frac{f(x)}{x-\alpha_1} = x^{n-1}+(\alpha_1+p_1)x^{n-2}+(\alpha_1^2+p_1\alpha_1+p_2)^{n-3}$$
$$+\cdots+(\alpha_1^{n-1}+p_1\alpha_1^{n-2}+\cdots+p_{n-1}),$$

$$\frac{f(x)}{x-\alpha_2} = x^{n-1}+(\alpha_2+p_1)x^{n-2}+(\alpha_2^2+p_1\alpha_2+p_2)^{n-3}$$
$$+\cdots+(\alpha_2^{n-1}+p_1\alpha_2^{n-2}+\cdots+p_{n-1}),$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\frac{f(x)}{x-\alpha_n} = x^{n-1}+(\alpha_n+p_1)x^{n-2}+(\alpha_n^2+p_1\alpha_n+p_2)^{n-3}$$
$$+\cdots+(\alpha_n^{n-1}+p_1\alpha_n^{n-2}+\cdots+p_{n-1})$$

Adding all the above n equations, we get,

$$f'(x) = nx^{n-1}+(S_1+np_1)x^{n-2}+(S_2+p_1S_1+np_2)^{n-3}$$
$$+\cdots+(S_{n-1}+p_1S_{n-2}+\cdots+np_{n-1}) \text{ -------- } (7.2)$$

But from (7.1),

$$f'(x) = nx^{n-1}+(n-1)p_1x^{n-2}+(n-2)p_2x^{n-3}+\cdots+p_{n-1} \text{ -------- } (7.3)$$

Equating the coefficients of like powers of x in (7.2) and in (7.3), we get,

$$S_1+p_1=0$$

$$S_2+p_1S_1+2p_2=0$$

$$S_3+p_1S_2+p_2S_1+3p_3=0$$

$$S_4+p_1S_3+p_2S_2+p_3S_1+4p_4=0$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$S_{n-1}+p_1S_{n-2}+p_2S_{n-3}+\cdots+p_{n-2}S_1+(n-1)p_{n-1}=0$$

Space for Hints

From these $n-1$ relations we can conclude the succession values of $S_1, S_2, S_3, \cdots, S_{n-1}$ in terms of the coefficients $p_1, p_2, p_3, \cdots, p_{n-1}$.

Now we shall extend this idea for the sums of all positive powers of the roots, namely, $S_n, S_{n+1}, S_{n+2}, \cdots, S_r$ where $r > n$.

We have $x^{r-n} f(x) = x^r + p_1 x^{r-1} + p_2 x^{r-2} + \cdots + p_n x^{r-n}$

Replacing in this identity, x by the roots $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_{n-1}$, in succession and adding, we get,

$S_n + p_1 S_{n-1} + p_2 S_{n-2} + \cdots + p_{n-1} S_2 + np_n = 0$

$S_{n+1} + p_1 S_n + p_2 S_{n-1} + \cdots + nS_1 = 0$

$S_{n+2} + p_1 S_{n+1} + p_2 S_n + \cdots + nS_2 = 0$ and so on.

Thus $S_r + p_1 S_{r-1} + p_2 S_{r-2} + \cdots + rp_r = 0$ if $r < n$

$S_r + p_1 S_{r-1} + p_2 S_{r-2} + \cdots + p_n S_{r-n} = 0$ if $r \geq n$

Note : To find the sum of the negative integral powers of the roots of the function $f(x) = 0$, put $x = \dfrac{1}{y}$ and find the sums of the corresponding positive powers of the transformed equation.

Example 7.2.1 :

If α, β, γ are the roots of the cubic equation $x^3 + px^2 + qx + r = 0$, find the value of $\alpha^3 + \beta^3 + \gamma^3$

Proof : Given that α, β, γ are the roots of the cubic equation $x^3 + px^2 + qx + r = 0$.

To find $\sum \alpha^3$

It is enough to find S_3

We krow that $S_3 + p_1 S_2 + p_2 S_1 + 3p_3 = 0$

$$S_2 + p_1 S_1 + 2p_2 = 0$$

$$S_1 + p_1 = 0$$

(i.e) $S_3 + pS_2 + qS_1 + 3r = 0$ -- (7.4)

$S_2 + pS_1 + 2q = 0$ -- (7.5)

Space for Hints

$$S_1 + p = 0 \text{ ------------------------------------ (7.6)}$$

From (7.6), $S_1 = -p$ -- (7.7)

From (7.5), $S_2 + p(-p) + 2q = 0$

(i.e) $S_2 = p^2 - 2q$ -- (7.8)

Using (7.4), (7.7) and (7.8), we get,

$S_3 + p(p^2 - 2q) + q(-p) + 3r = 0$

(i.e) $S_3 + p(p^2 - 2q) + q(-p) + 3r = 0$

(i.e) $S_3 + p^3 - 2pq - pq + 3r = 0$

(i.e) $S_3 = 3pq - p^3 - 3r$

Thus $\sum \alpha^3 = S_3 = 3pq - p^3 - 3r$.

Example 7.2.2 :

Calculate the sum of the cubes of the roots of the equation $x^4 + 2x + 3 = 0$

Proof : Given that $x^4 + 2x + 3 = 0$

Compare the given equation with $x^4 + p_1 x^3 + p_2 x^2 + p_3 x + p_4 = 0$, we have, $p_1 = 0$, $p_2 = 0$, $p_3 = 2$, $p_4 = 3$.

To find $\sum \alpha^3$

It is enough to find S_3

We know that $S_3 + p_1 S_2 + p_2 S_1 + 3p_3 = 0$ -------------------------- (7.9)

$S_2 + p_1 S_1 + 2p_2 = 0$ -------------------------- (7.10)

$S_1 + p_1 = 0$ -------------------------- (7.11)

From (7.11), $S_1 + (0) = 0$

(i.e) $S_1 = 0$ -- (7.12)

From (7.10), $S_2 + (0)S_1 + 2(0) = 0$

(i.e) $S_2 = 0$ -- (7.13)

From (7.9), $S_3 + (0)S_2 + (0)S_1 + 3(2) = 0$

(i.e) $S_3 = -6$ Hence $\sum \alpha^3 = -6$

Example 7.2.3 :

Calculate the sum of the cubes of the roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$

Proof : Given that $x^3 - 6x^2 + 11x - 6 = 0$

Compare the given equation with $x^3 + p_1 x^2 + p_2 x + p_3 = 0$, we have,

$p_1 = -6,\ p_2 = 11,\ p_3 = -6$.

To find $\sum \alpha^3$

It is enough to find S_3

We know that $S_3 + p_1 S_2 + p_2 S_1 + 3p_3 = 0$ -------------------------- (7.14)

$S_2 + p_1 S_1 + 2p_2 = 0$ ---------------------------- (7.15)

$S_1 + p_1 = 0$ -------------------------- (7.16)

From (7.16), $S_1 + (-6) = 0$

(i.e) $S_1 = 6$ -- (7.17)

From (7.15), $S_2 + (-6)6 + 2(11) = 0$ (using (7.17))

(i.e) $S_2 - 36 + 22 = 0$

(i.e) $S_2 = 14$ -- (7.18)

From (7.14), $S_3 + (-6)(14) + (11)(6) + 3(-6) = 0$ (using (7.17) and (7.18))

(i.e) $S_3 - 84 + 66 - 18 = 0$

(i.e) $S_3 = 36$

Hence $\sum \alpha^3 = 36$

Example 7.2.4 :

Show that the sum of the fourth powers of the roots of the equation $x^5 + px^3 + qx^2 + s = 0$ is $2p^2$

Proof : Given that $x^5 + px^3 + qx^2 + s = 0$

Compare the given equation with $x^5 + p_1 x^4 + p_2 x^3 + p_3 x^2 + p_4 x + p_5 = 0$, we have,

$p_1 = 0,\ p_2 = p,\ p_3 = q,\ p_4 = 0,\ p_5 = s$.

Space for Hints

Space for Hints

To prove that $\sum \alpha^4 = 0$

It is enough to prove that $S_4 = 0$

We know that $S_4 + p_1S_3 + p_2S_2 + p_3S_1 + 4p_4 = 0$ ---------------- (7.19)

$S_3 + p_1S_2 + p_2S_1 + 3p_3 = 0$ --------------- (7.20)

$S_2 + p_1S_1 + 2p_2 = 0$ --------------- (7.21)

$S_1 + p_1 = 0$ --------------- (7.22)

From (7.22), $S_1 = 0$

From (7.21), $S_2 + p(0) + 2p = 0$

(i.e) $S_2 = -2p$

From (7.20), $S_3 + (0) + (0) + 3q = 0$

(i.e) $S_3 = -3q$

Again from (7.19), $S_4 + 0 + p(-2p) + 0 + 0 = 0$

(i.e) $S_4 = 2p^2$

Thus the sum of the fourth powers of the roots of the equation $x^5 + px^3 + qx^2 + s = 0$ is $2p^2$.

This proves the problem.

Example 7.2.5 :

If α, β, γ are the roots of $x^3 + qx + r = 0$ then prove that (i) $3S_2S_5 = 5S_3S_4$

and (ii) $\frac{\sum \alpha^5}{5} = \frac{\sum \alpha^3}{3} \cdot \frac{\sum \alpha^2}{2}$

Proof Given that α, β, γ are the roots of $x^3 + qx + r = 0$

Compare the given equation with $x^3 + p_1 x^2 + p_2 x + p_3 = 0$, we have,

$p_1 = 0$, $p_2 = q$, $p_3 = r$.

Now $S_5 + p_1S_4 + p_2S_3 + p_3S_2 = 0$

(i.e) $S_5 + qS_3 + rS_2 = 0$ -- (7.23)

Now $S_3 + p_1S_2 + p_2S_1 + 3p_3 = 0$

(i.e) $S_3 + qS_1 + 3r = 0$ -- (7.24)

Now $S_2 + p_1 S_1 + 2p_2 = 0$

(i.e) $S_2 + 2q = 0$

(i.e) $S_2 = -2q$ -- (7.25)

and $S_1 + p_1 = 0$

(i.e) $S_1 = 0$ -- (7.26)

From (7.24) and (7.26), $S_3 + 3r = 0$

(i.e) $S_3 = -3r$ (7.27)

Using (7.23), (7.27) and (7.25), we have, $S_5 + q(-3r) + r(-2q) = 0$

(i.e) $S_5 = 5qr$

$$\Rightarrow \frac{S_5}{5} = qr$$

$$\Rightarrow \frac{S_5}{5} = \left(\frac{S_2}{-2}\right) \cdot \left(\frac{S_3}{-3}\right) \text{ (using (7.25) and (7.27))}$$

$$\Rightarrow \frac{S_5}{5} = \frac{S_2}{2} \cdot \frac{S_3}{3}$$

This proves (ii)

Again, we know that $S_4 + p_1 S_3 + p_2 S_2 + p_3 S_1 = 0$

(i.e) $S_4 + q(-2q) = 0$ (from (7.25), (7.26) and (7.27))

(i.e) $S_4 = 2q^2$

Now $3S_2S_5 = 3(-2q)(5qr) = -30q^2r$ ------------------------------- (7.28)

and $5S_3S_4 = 5(-3r)(2q^2) = -30q^2r$ -------------------------------(7.29)

Thus from (7.28) and (7.29), we have, $3S_2S_5 = 5S_3S_4$

This proves (i)

Hence the problem.

Example 7.2.6 :

If α, β, γ are the roots of $x^3 - 7x + 7 = 0$ then find $\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4}$

Proof : Given that α, β, γ are the roots of $x^3 - 7x + 7 = 0$ --------- (7.30)

Space for Hints

Space for Hints

Put $x = \frac{1}{y}$

$$\therefore (7.30) \Rightarrow \left(\frac{1}{y}\right)^3 - 7\left(\frac{1}{y}\right) + 7 = 0$$

$$\Rightarrow 7y^3 - 7y^2 + 1 = 0$$

Compare the given equation with $y^3 + p_1 y^2 + p_2 y + p_3 = 0$, we have,

$p_1 = -1, \quad p_2 = 0, \quad p_3 = 1.$

Now $S_4 + p_1 S_3 + p_2 S_2 + p_3 S_1 = 0$ ---------------------------- (7.31)

$S_3 + p_1 S_2 + p_2 S_1 + 3p_3 = 0$ ---------------------------- (7.32)

$S_2 + p_1 S_1 + 2p_2 = 0$ ---------------------------- (7.33)

$S_1 + p_1 = 0$ ---------------------------- (7.34)

From (7.34), $S_1 - 1 = 0$

(i.e) $S_1 = 1$ -- (7.35)

From (7.33), $S_2 + (1)(-1) + 0 = 0$

(i.e) $S_2 = 1$ -- (7.36)

From (7.32), $S_3 + (-1)(1) + 0 + 3\left(\frac{1}{7}\right) = 0$

(i.e) $S_3 = \frac{4}{7}$ -- (7.37)

From (7.31), $S_4 + (-1)\left(\frac{4}{7}\right) + 0 + \left(\frac{1}{7}\right)(1) = 0$

(i.e) $S_4 = \frac{3}{7}$

Thus $\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} = S_4 = \frac{3}{7}$

7.3. Transformation of equations

In order to solve a given equation $f(x) = 0$ we want to transform into another whose roots bear a certain assigned relation with those of the given

equation and whose solution may be mort easier than that of the given equation. Having solved the transformed equation we can able to find the roots of the given equation from the given relation between the roots of the given equation.

Space for Hints

Example 7.3.1 :

Find the equation whose roots are the roots of $x^5+6x^4+6x^3-7x^2+2x-1=0$ with the sign changed.

Proof : Given that $x^5+6x^4+6x^3-7x^2+2x-1=0$ ------- (7.38)

Put $x=-y$ in (7.38) , we get, $-x^5+6x^4-6x^3-7x^2-2x-1=0$

(i.e) $x^5+6x^4+6x^3+7x^2+2x+1=0$ which is the required equation.

Example 7.3.2 :

Transform the equation $3x^3+4x^2+5x-6=0$ into one in which the coefficient of x^3 is unity and all coefficients are integral.

Proof : Given that $3x^3+4x^2+5x-6=0$ (7.39)

Multiply the roots of (7.39) by 3, we get, the transformed equation as

$3x^3+(3)(4)x^2+(3)^2(5)x-(3)^3(6)=0$

Dividing 3 on both sides of the above equation, we get,

$3x^3+4x^2+15x-54=0$

Example 7.3.3 :

Remove the fractional coefficients from the equation

$$2x^3+\frac{3}{2}x^2-\frac{1}{8}x-\frac{3}{16}=0$$

Proof : Given that $2x^3+\frac{3}{2}x^2-\frac{1}{8}x-\frac{3}{16}=0$ (7.40)

Multiply the roots of (7.40) by 4, we get, the transformed equation as

$$2x^3+(4)\times\frac{3}{2}x^2-(4)^2\times\frac{1}{8}x-(4)^3\times\frac{3}{16}=0$$

(i.e) $2x^3+6x^2-2x-12=0$

Space for Hints

Dividing 2 on both sides of the above equation, we get,

$x^3 + 3x^2 - x - 6 = 0$ which is the required transformed equation.

Check your progress

Questions :

(1) Change the equation $2x^4 - 3x^3 + 3x^2 - x + 2 = 0$ into another the coefficients of whose highest term will be unity.

(2) Remove the fractional coefficients from the equation

$$x^3 - \frac{1}{4}x^2 + \frac{1}{3}x - 1 = 0$$

(3) Remove the fractional coefficients from the equation

$$x^3 + \frac{3}{2}x^2 + \frac{5}{8}x + \frac{1}{108} = 0$$

7.3. Transformation of equations

Question : Find the transformed equation whose roots are the reciprocal of the roots of the given equation.

Solution : Let $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$ be the roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_n = 0 \qquad (7.41)$$

Now $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_n$

$$= (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n) \qquad (7.42)$$

put $x = \dfrac{1}{y}$ in (7.42), we have,

$$\left(\frac{1}{y}\right)^n + p_1\left(\frac{1}{y}\right)^{n-1} + p_2\left(\frac{1}{y}\right)^{n-2} + \cdots + p_n$$

$$= \left(\frac{1}{y} - \alpha_1\right)\left(\frac{1}{y} - \alpha_2\right)\left(\frac{1}{y} - \alpha_3\right)\cdots\left(\frac{1}{y} - \alpha_n\right)$$

Multiplying by y^n on both sides of the equation, we get,

$$p_n y^n + p_{n-1} y^{n-1} + p_{n-2} y^{n-2} + \cdots + p_1 y + 1$$

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$$=(\alpha_1\alpha_2\alpha_3 \text{L } \alpha_n)\left(\frac{1}{\alpha_1}-y\right)\left(\frac{1}{\alpha_2}-y\right)\left(\frac{1}{\alpha_3}-y\right)\text{L }\left(\frac{1}{\alpha_n}-y\right)$$

Hence, the equation $p_n y^n + p_{n-1} y^{n-1} + p_{n-2} y^{n-2} + \text{L } + p_1 y + 1 = 0$ has roots $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3}, \text{L }, \frac{1}{\alpha_n}$.

7.4. Reciprocal roots and Reciprocal equation

Definition : An equation $f(x)=0$ is called a reciprocal equation if whenever α is a root of the equation then $\frac{1}{\alpha}$ is also a root of the equation.

Theorem 7.4.1 :

An equation $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \text{L } + a_n = 0$ is a reciprocal equation if and only if $a_{n-r} = \pm a_r$, $(0 \le r \le n)$

Proof : Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \text{L } + a_n = 0$ be a reciprocal equation.

$$\therefore\ f\left(\frac{1}{x}\right) = a_0\left(\frac{1}{x}\right)^n + a_1\left(\frac{1}{x}\right)^{n-1} + a_2\left(\frac{1}{x}\right)^{n-2} + \text{L } + a_n$$

$$\Rightarrow\ x^n f\left(\frac{1}{x}\right) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \text{L } + a_0$$

Now $f(x)=0$ is a reciprocal equation, then $f(x)=0$ and $x^n f\left(\frac{1}{x}\right)=0$ both have the same roots and the corresponding coefficients of the two equations are proportional.

$$\therefore\ \frac{a_0}{a_n} = \frac{a_1}{a_{n-1}} = \frac{a_2}{a_{n-2}} = \text{L } = \frac{a_n}{a_0} = k \text{ (say)}$$

Now $\frac{a_0}{a_n} = \frac{a_n}{a_0} = k$

$$\Rightarrow\ k \cdot k = \frac{a_0}{a_n}\cdot\frac{a_n}{a_0}$$

Space for Hints

$\Rightarrow k^2 = 1$

$\Rightarrow k = \pm 1$

Thus $\frac{a_r}{a_{n-r}} = \pm 1$

(i.e) $a_{n-r} = \pm a_r$

Conversely, let $a_{n-r} = \pm a_r$.

Then the equations $f(x) = 0$ and $x^n f\left(\frac{1}{x}\right) = 0$ are same equations and hence $f(x) = 0$ is reciprocal equation.

This proves the theorem.

Definition :

A reciprocal equation $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n = 0$ is said to be reciprocal equation of ***first type*** if $a_{n-r} = a_r$ and is said to be of ***second type*** if $a_{n-r} = -a_r$.

Note :

(1) If $f(x) = 0$ is reciprocal equation of first type and odd degree then $x+1$ is a factor of $f(x)$ and $\frac{f(x)}{x+1}$ is a standard reciprocal equation

(2) If $f(x) = 0$ is reciprocal equation of second type and odd degree then $x-1$ is a factor of $f(x)$ and $\frac{f(x)}{x-1}$ is a standard reciprocal equation

(3) If $f(x) = 0$ is reciprocal equation of second type and even degree then $x^2 - 1$ is a factor of $f(x)$ and $\frac{f(x)}{x^2-1}$ is a standard reciprocal equation

Example 7.4.1 :

Solve $60x^4 - 736x^3 + 1433x^2 - 736x + 60 = 0$

Proof : Given that $60x^4 - 736x^3 + 1433x^2 - 736x + 60 = 0$ --------(7.43)

Clearly (7.43) is reciprocal equation.

Space for Hints

Dividing the equation (7.43) by x^2, we get,

$$60x^2 - 736x + 1433 - 736\frac{1}{x} + 60\frac{1}{x^2} = 0$$

(i.e) $60\left(x^2 + \frac{1}{x^2}\right) - 736\left(x + \frac{1}{x}\right) + 1433 = 0$ ---------------------- (7.44)

let $y = x + \frac{1}{x}$

$\therefore\ y^2 - 2 = x^2 + \frac{1}{x^2}$

Thus (7.44) $\Rightarrow\ 60y^2 - 736y + 1433 = 0$

$\therefore\ y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(i.e) $y = \frac{736 \pm \sqrt{541696 - 315120}}{120}$

(i.e) $y = \frac{736 \pm 476}{120}$

(i.e) $y = \frac{101}{10}$ or (i.e) $y = \frac{13}{6}$

When $y = \frac{101}{10}$ then $x + \frac{1}{x} = \frac{101}{10}$

(i.e) $10x^2 + 1 = 101x$

(i.e) $10x^2 - 101x + 1 = 0$

(i.e) $(10x - 1)(x - 10) = 0$

(i.e) $10x - 1 = 0$ or $x - 10 = 0$

(i.e) $x = \frac{1}{10}$ or $x = 10$

When $y = \frac{13}{6}$ then $x + \frac{1}{x} = \frac{13}{6}$

(i.e) $6x^2 + 6 = 13x$

(i.e) $6x^2 - 13x + 6 = 0$

(i.e) $(2x - 3)(3x - 2) = 0$

(i.e) $2x - 3 = 0$ or $3x - 2 = 0$

Space for Hints

(i.e) $x=\frac{3}{2}$ or $x=\frac{2}{3}$

Thus the roots of the equation (7.43) are $10, \frac{1}{10}, \frac{3}{2}, \frac{2}{3}$.

Example 7.4.2 :

Solve $x^5+x^4+x^3+x^2+x+1=0$

Proof : Given that $x^5+x^4+x^3+x^2+x+1=0$ ------------------ (7.45)

Clearly (7.45) is of first type and odd degree.

$\therefore$ $x+1$ is a factor of (7.45).

Dividing the equation (7.45) by $x=-1$ synthetically, we get,

$$\begin{array}{r|rrrrr|r} -1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 0 & -1 & 0 & -1 & 0 & -1 \\ \hline & 1 & 0 & 1 & 0 & 1 & 0 \end{array}$$

Thus $x^5+x^4+x^3+x^2+x+1 \equiv (x+1)(x^4+0x^3+x^2+0x+1)$

(i.e) $x^5+x^4+x^3+x^2+x+1 \equiv (x+1)(x^4+x^2+1)$

Now $x^5+x^4+x^3+x^2+x+1=0$

Then $(x+1)(x^4+x^2+1)=0$

(i.e) $x+1=0$ or $x^4+x^2+1=0$

(i.e) $x=-1$ or $x^4+x^2+1=0$

Now we shall solve $x^4+x^2+1=0$ ------------------------------------ (7.46)

Dividing the equation (7.46) by x^2, we get,

$$x^2+1+\frac{1}{x^2}=0$$

(i.e) $\left(x^2+\frac{1}{x^2}\right)+1=0$ -- (7.47)

let $y=x+\frac{1}{x}$

$\therefore\ y^2 - 2 = x^2 + \dfrac{1}{x^2}$

Thus (7.47) $\Rightarrow\ y^2 - 2 + 1 = 0$

(i.e) $y^2 = 1$

$\therefore\ y = \pm 1$

When $y = 1$ then $x + \dfrac{1}{x} = 1$

(i.e) $x^2 + 1 = x$

(i.e) $x^2 - x + 1 = 0$

$\therefore\ x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(i.e) $x = \dfrac{1 \pm \sqrt{1-4}}{2}$

(i.e) $x = \dfrac{1 \pm i\sqrt{3}}{2}$

When $y = -1$ then $x + \dfrac{1}{x} = -1$

(i.e) $x^2 + 1 = -x$

(i.e) $x^2 + x + 1 = 0$

$\therefore\ x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(i.e) $x = \dfrac{-1 \pm \sqrt{1-4}}{2}$

(i.e) $x = \dfrac{-1 \pm i\sqrt{3}}{2}$

Thus the roots of the equation (7.43) are $-1, \dfrac{\pm 1 \pm i\sqrt{3}}{2}$.

Example 7.4.3 :

Solve $x^5 + 8x^4 + 21x^3 + 21x^2 + 8x + 1 = 0$

Proof : Given that $x^5 + 8x^4 + 21x^3 + 21x^2 + 8x + 1 = 0$ -------------- (7.48)

Clearly (7.48) is of first type and odd degree.

Space for Hints

Space for Hints

$\therefore\ x+1$ is a factor of (7.45).

Dividing the equation (7.45) by $x=-1$ synthetically, we get,

$$\begin{array}{r|rrrrrr} -1 & 1 & 8 & 21 & 21 & 8 & 1 \\ & 0 & -1 & -7 & -14 & -7 & -1 \\ \hline & 1 & 7 & 14 & 7 & 1 & 0 \end{array}$$

Thus $x^5+8x^4+21x^3+21x^2+8x+1 \equiv (x+1)(x^4+7x^3+14x^2+7x+1)$

Now $x^5+x^4+x^3+x^2+x+1=0$

Then $(x+1)(x^4+7x^3+14x^2+7x+1)=0$

(i.e) $x+1=0$ or $x^4+7x^3+14x^2+7x+1=0$

(i.e) $x=-1$ or $x^4+7x^3+14x^2+7x+1=0$

Now we shall solve $x^4+7x^3+14x^2+7x+1=0$ ---------- (7.49)

Dividing the equation (7.46) by x^2, we get,

$$x^2+7x+14+7\frac{1}{x}+\frac{1}{x^2}=0$$

(i.e) $\left(x^2+\frac{1}{x^2}\right)+7\left(x+\frac{1}{x}\right)+14=0$ -------------------------- (7.50)

let $y=x+\frac{1}{x}$

$\therefore\ y^2-2=x^2+\frac{1}{x^2}$

Thus (7.50) $\Rightarrow$ $y^2-2+7y+14=0$

(i.e) $y^2+7y+12=0$

(i.e) $(y+3)(y+4)=0$

(i.e) $y+3=0$ or $y+4=0$

$\therefore\ y=-3$ or $y=-4$

When $y=-3$ then $x+\frac{1}{x}=-3$

(i.e) $x^2+1=-3x$

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(i.e) $x^2 + 3x + 1 = 0$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(i.e) $x = \dfrac{-3 \pm \sqrt{5}}{2}$

When $y = -4$ then $x + \dfrac{1}{x} = -4$

(i.e) $x^2 + 1 = -4x$

(i.e) $x^2 + 4x + 1 = 0$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(i.e) $x = \dfrac{-4 \pm \sqrt{16 - 4}}{2}$

(i.e) $x = \dfrac{-4 \pm \sqrt{12}}{2}$

(i.e) $x = \dfrac{-4 \pm 2\sqrt{3}}{2}$

(i.e) $x = -2 \pm \sqrt{3}$

Thus the roots of the equation (7.43) are $\dfrac{-3 \pm \sqrt{5}}{2}, -2 \pm \sqrt{3}$.

Example 7.4.4 :

Solve $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0$

Proof : Given that $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0$ ------- (7.51)

Clearly (7.51) is of second type and even degree.

$\therefore$ $x^2 - 1$ is a factor of (7.51).

Dividing the equation (7.51) by $x = 1$ and $x = -1$ synthetically, we get,

Space for Hints

1	6	−25	31	0	−31	25	−6
	0	6	−19	12	12	−19	6
−1	6	−19	12	12	−19	6	0
	0	−6	25	−37	25	−6	
	6	−25	37	−25	6	0	

Thus $6x^6-25x^5+31x^4-31x^2+25x-6$

$$\equiv (x^2-1)(6x^4-25x^3+37x^2-25x+6)$$

Now $6x^6-25x^5+31x^4-31x^2+25x-6=0$

Then $(x^2-1)(6x^4-25x^3+37x^2-25x+6)=0$

(i.e) $x^2-1=0$ or $6x^4-25x^3+37x^2-25x+6=0$

(i.e) $x=\pm 1$ or $6x^4-25x^3+37x^2-25x+6=0$

Now we shall solve $6x^4-25x^3+37x^2-25x+6=0$ -------------- (7.52)

Dividing the equation (7.52) by x^2, we get,

$$6x^2-25x+37-25\frac{1}{x}+6\frac{1}{x^2}=0$$

(i.e) $6\left(x^2+\frac{1}{x^2}\right)-25\left(x+\frac{1}{x}\right)+37=0$ ------------------------------ (7.53)

let $y=x+\frac{1}{x}$

$\therefore\ y^2-2=x^2+\frac{1}{x^2}$

Thus (7.53) $\Rightarrow$ $y^2-2-25y+37=0$

(i.e) $y^2-25y+35=0$

(i.e) $(2y-5)(3y-5)=0$

(i.e) $2y-5=0$ or $3y-5=0$

$\therefore\ y=\frac{5}{2}$ or $y=\frac{5}{3}$

When $y=\frac{5}{2}$ then $x+\frac{1}{x}=\frac{5}{2}$

(i.e) $2x^2+1=5x$

(i.e) $2x^2-5x+1=0$

(i.e) $(2x-1)(x-2)=0$

(i.e) $2x-1=0$ or $x-2=0$

(i.e) $x=\frac{1}{2}$ or $x=2$

When $y=\frac{5}{3}$ then $x+\frac{1}{x}=\frac{5}{3}$

(i.e) $3x^2+1=5x$

(i.e) $3x^2-5x+1=0$

$\therefore\ x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

(i.e) $x=\dfrac{5\pm i\sqrt{11}}{6}$

Thus the roots of the equation (7.51) are $\pm1, 2, \frac{1}{2}, \frac{5\pm i\sqrt{11}}{6}$.

Check your progress

Questions :

Solve the following equations

(1) $4x^4-20x^3+33x^2-20x+4=0$

(2) $2x^6-9x^5+10x^4-3x^3+10x^2-9x+2=0$

(3) $2x^5+x^4+x+2=12x^2(x+1)$

(4) $x^5-5x^3+5x^2-1=0$

(5) $x^{10}-3x^8+5x^6-5x^4+3x^2-1=0$

(Answers :

(1) $2, \frac{1}{2}, 2, \frac{1}{2}$

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Space for Hints

(2) $2, \frac{1}{2}, \frac{3\pm\sqrt{5}}{2}, \frac{-1\pm i\sqrt{3}}{2}$

(3) $-1, -2, -\frac{1}{2}, \frac{3\pm\sqrt{5}}{2}$

(4) $1, 1, 1, \frac{-3\pm\sqrt{5}}{2}$

(5) $\pm 1, \frac{\pm\sqrt{3}\pm i}{2}, \frac{\pm\sqrt{3}\pm i}{2}$

7.5. Increasing and decreasing the roots

Let the roots of the equation $f(x) \equiv a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n = 0$ be $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$ and suppose we want to find an equation whose roots are $\alpha_1 - h, \alpha_2 - h, \alpha_3 - h, \cdots, \alpha_n - h$.

Now $f(x) \equiv a_0 (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n)$ --------------- (7.54)

Let $x = y + h$ then (7.54) changes as

$f(y+h) \equiv a_0 (y+h-\alpha_1)(y+h-\alpha_2)\cdots(y+h-\alpha_n)$ --------------- (7.55)

The right hand side of (7.55) vanishes when $y = \alpha_r - h$ for $r = 1, 2, 3, \cdots, n$.

Hence if an equation is to be transformed into another whose roots are those of the first diminished by *h*, simply substitute *x* by $y + h$ in the given equation. Then we get the transformed equation as

$a_0 (y+h)^n + a_1 (y+h)^{n-1} + a_2 (y+h)^{n-2} + \cdots + a_n = 0$ ----------(7.56)

Equation (7.56) can be written as

$A_0 y^n + A_1 y^{n-1} + A_2 y^{n-2} + \cdots + A_{n-1} y + A_n = 0$ -------------------- (7.57)

Here $A_1, A_2, A_3, \cdots, A_n$ are functions of $a_1, a_2, a_3, \cdots, a_n$

If we take $y = x - h$ and substituting in equation (7.57), we the original equation.

Example 7.5.1 :

Diminish the roots of $x^4 - 8x^3 - 19x^2 - 12x + 2 = 0$ by 2.

Solution : Given that $x^4 - 8x^3 - 19x^2 - 12x + 2 = 0$

The operations of diminishing the roots by 2 are shown below :

2	1	-8	19	-12	2
	0	2	-12	14	4
	1	-6	7	2	6
	0	2	-8	-2	
	1	-4	-1	0	
	0	2	-4		
	1	-2	-5		
	0	2			
	1	0			

$\therefore$ the transformed equation is $x^4 - 5x^2 + 6 = 0$

Example 7.5.2 :

Diminish by 2 the roots of the equation $x^5 - 3x^4 - 2x^3 + 15x^2 + 20x - 15 = 0$

Solution : Given that $x^5 - 3x^4 - 2x^3 + 15x^2 + 20x - 15 = 0$

The operations of diminishing the roots by 2 are shown below :

2	1	-3	-2	15	20	-15
	0	2	-2	-8	14	68
	1	-1	-4	7	34	53
	0	2	2	-4	6	
	1	1	-2	3	40	
	0	2	6	8		
	1	3	4	11		
	0	2	10			
	1	5	14			
	0	2				
	1	7				

$\therefore$ the transformed equation is $x^5 + 7x^4 + 14x^3 + 11x^2 + 40x + 53 = 0$

Space for Hints

Space for Hints

Example 7.5.3 :

Find the equation whose roots are the roots of the equation $4x^5-2x^3+7x-3=0$ each increased by 2

Solution : Given that $4x^5-2x^3+7x-3=0$

The operations of diminishing the roots by 2 are shown below :

-2	4	0	-2	0	7	-3
	0	-8	16	-28	56	-126
	4	-8	14	-28	63	$\boxed{-129}$
	0	-8	32	-92	240	
	4	-16	46	-120	$\boxed{303}$	
	0	-8	48	-188		
	4	-24	94	$\boxed{-308}$		
	0	-8	64			
	4	-32	$\boxed{158}$			
	0	-8				
	4	$\boxed{-40}$				

$\therefore$ the transformed equation is $4x^5-40x^4+158x^3-308x^2+303x-129=0$

Example 7.5.4 :

If α,β,γ are the roots of $8x^3-4x^2+6x-1=0$, find the equation whose roots are $\alpha+\frac{1}{2},\beta+\frac{1}{2},\gamma+\frac{1}{2}$.

Solution : Given that α,β,γ are the roots of $8x^3-4x^2+6x-1=0$

We shall find the equation whose roots are $\alpha+\frac{1}{2},\beta+\frac{1}{2},\gamma+\frac{1}{2}$.

It is enough to find the equation whose roots are increased by $\frac{1}{2}$ of the roots of the given equation.

The operations of increasing roots by $\frac{1}{2}$ are shown below :

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- 1/2	8	-4	6	-1
	0	-4	4	-5
	8	-8	10	-6
	0	-4	6	
	8	-12	16	
	0	-4		
	8	-16		

$\therefore$ the transformed equation is $8x^3 - 16x^2 + 16x - 6 = 0$

(i.e) $4x^3 - 8x^2 + 8x - 3 = 0$ which is the required equation

Example 7.5.5 :

Show that the equation $x^4 - 10x^3 + 23x^2 - 6x - 15 = 0$ can be transformed into a reciprocal equation by diminishing the roots by 2. Hence or otherwise, solve the equation.

Solution : Given that $x^4 - 10x^3 + 23x^2 - 6x - 15 = 0$ --------------- (7.58)

The operations of diminishing the roots of (7.58) by 2 are shown below :

2	1	-10	23	-6	-15
	0	2	-16	14	16
	1	-8	7	8	1
	0	2	-12	-10	
	1	-6	-5	-2	
	0	2	-8		
	1	-4	-13		
	0	2			
	1	-2			

The transformed equation is $x^4 - 2x^3 - 13x^2 - 2x + 1 = 0$ ---------- (7.59)

In (7.59), $a_{n-r} = a_r$ for $0 \le r \le 4$, and $\therefore$ (7.59) is a reciprocal equation.

Dividing x^2 on both sides of (7.59), we get,

$$x^2 - 2x - 13 - 2\frac{1}{x} + \frac{1}{x^2} = 0$$

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$$\text{(i.e)} \left(x^2+\frac{1}{x^2}\right)-2\left(x+\frac{1}{x}\right)-13=0 \quad \text{------------------------------------} \quad (7.60)$$

Let $y=x+\dfrac{1}{x}$

$\therefore \; y^2-2=x^2+\dfrac{1}{x^2}$

Thus (7.60) $\Rightarrow \; y^2-2-2y-13=0$

$\Rightarrow \; y^2-2y-15=0$

$\Rightarrow \; (y-5)(y+3)=0$

$\Rightarrow \; y-5=0$ or $y+3=0$

$\Rightarrow \; y=5$ or $y=-3$

When $y=5$, then $x+\dfrac{1}{x}=5$

(i.e) $x^2+1=5x$

(i.e) $x^2-5x+1=0$

$$\therefore \; x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\text{(i.e)} \; x=\frac{5\pm\sqrt{1-4}}{2}$$

$$\text{(i.e)} \; x=\frac{5\pm i\sqrt{3}}{2}$$

When $y=-3$, then $x+\dfrac{1}{x}=-3$

(i.e) $x^2+1=-3x$

(i.e) $x^2+3x+1=0$

$$\therefore \; x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\text{(i.e)} \; x=\frac{-3\pm\sqrt{9-4}}{2}$$

$$\text{(i.e)} \; x=\frac{-3\pm\sqrt{5}}{2}$$

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Hence the roots of the transformed equation are $\frac{5 \pm i\sqrt{3}}{2}, \frac{-3 \pm \sqrt{5}}{.2}$.

and the roots of the given equation are $\frac{5 \pm i\sqrt{3}}{2} + 2, \frac{-3 \pm \sqrt{5}}{2} + 2$

(i.e) $\frac{9 \pm i\sqrt{3}}{2}, \frac{1 \pm \sqrt{5}}{2}$

Example 7.5.6 :

Find the equation whose roots exceed by 2 the roots of the equation

$4x^4 + 32x^3 + 83x^2 + 76x + 21 = 0$.

Solution : Given that $4x^4 + 32x^3 + 83x^2 + 76x + 21 = 0$ ------------ (7.61)

The operations of diminishing the roots of (7.61) by 2 are shown below :

-2	4	32	83	76	21
	0	-8	-48	-70	-12
	4	24	35	6	9
	0	-8	-32	-6	
	4	16	3	0	
	0	-8	-16		
	4	8	-13		
	0	-8			
	4	0			

The transformed equation is $4x^4 - 13x^2 + 9 = 0$ ---------------------- (7.62)

Let $y = x^2$

Thus (7.62) $\Rightarrow 4y^2 - 13y + 9 = 0$

$\Rightarrow (y-1)(4y-9) = 0$

$\Rightarrow y - 1 = 0$ or $4y - 9 = 0$

$\Rightarrow y = 1$ or $y = \frac{9}{4}$

$\Rightarrow x^2 = 1$ or $x^2 = \frac{9}{4}$

$\Rightarrow x = \pm 1$ or $x = \pm \frac{3}{2}$

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$\therefore$ the roots of (7.62) are $\pm 1, \pm \frac{3}{2}$

Hence the roots of the given equation are $1 - 2, -1 - 2, \frac{3}{2} - 2, -\frac{3}{2} - 2$

(i.e) $-1, -3, -\frac{1}{2}, -\frac{7}{2}$.

Example 7.5.7 :

Solve the equation $x^4 + 4x^3 - 2x^2 - 12x - 3 = 0$ by transforming this equat into another whose roots are increased by unity.

Solution : Given that $x^4 + 4x^3 - 2x^2 - 12x - 3 = 0$ ------------------ (7.63)

The roots of (7.63) increased by 1 is same as the roots are diminished by –

The operations of diminishing the roots of (7.63) by -1 are shown below

-1	1	4	-2	-12	-3
	0	-1	-3	5	7
	1	3	-5	-7	4
	0	-1	-2	7	
	1	2	-7	0	
	0	-1	-1		
	1	1	-8		
	0	-1			
	1	0			

The transformed equation is $x^4 - 8x^2 + 4 = 0$ ------------------------ (7.6

Let $y = x^2$

Thus (7.64) $\Rightarrow$ $y^2 - 8y + 4 = 0$

$$\Rightarrow y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow y = \frac{8 \pm \sqrt{64 - 16}}{2}$$

$$\Rightarrow y = \frac{8 \pm \sqrt{48}}{2}$$

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$$\Rightarrow \ y=\frac{8\pm\sqrt{16\times 3}}{2}$$

$$\Rightarrow \ y=\frac{8\pm 4\sqrt{3}}{2}$$

$$\Rightarrow \ y=4\pm 2\sqrt{3}$$

$$\Rightarrow \ x^2=1+\left(\sqrt{3}\right)^2\pm 2\sqrt{3}$$

$$\Rightarrow \ x^2=\left(1\pm\sqrt{3}\right)^2$$

$$\Rightarrow \ x=\pm\left(1\pm\sqrt{3}\right)$$

$$\Rightarrow \ x=\left(1\pm\sqrt{3}\right) \text{ or } \Rightarrow \ x=-\left(1\pm\sqrt{3}\right)$$

$$\Rightarrow \ x=1\pm\sqrt{3} \text{ or } \Rightarrow \ x=-1\mathrm{m}\sqrt{3}$$

$$\Rightarrow \ x=1\pm\sqrt{3} \text{ or } \Rightarrow \ x=-1\pm\sqrt{3}$$

$\therefore$ the roots of (7.64) are $1\pm\sqrt{3}, -1\pm\sqrt{3}$

Hence the roots of the given equation are $1\pm\sqrt{3}-1, -1\pm\sqrt{3}-1$

(i.e) $1+\sqrt{3}-1, -1-\sqrt{3}-1, -1+\sqrt{3}-1, -1-\sqrt{3}-1$.

(i.e) $\sqrt{3}, -\sqrt{3}, -2+\sqrt{3}, -2-\sqrt{3}$

(i.e) $\pm\sqrt{3}, -2\pm\sqrt{3}$.

Check your progress

Questions :

(1) Diminish the roots of the equation $5x^3-13x^2-12x+7=0$ by 23

(Answer : $5x^3+332x^2+7325x+53689=0$)

(2) Find the quotient and remainder when $3x^3+8x^2+8x+12$ is divided by $x-4$

(Answer : Quotient : $3x^2+20x+88$ and remainder : 364)

(3) Find the quotient and remainder when $2x^6+3x^5-15x^2+2x-4$ is divided by $x+5$

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(Answer : Quotient : $2x^5-7x^4+35x^3-175x^2+860x-4298$ and remainder : 21486)

(4) Diminish the roots of the equation $x^4-5x^3+7x^2-4x+5=0$ by 2

(Answer : $x^4+3x^3+x^2-4x+1=0$)

(5) Increase by 7 the roots of the equation $3x^4+7x^3-15x^2+x-25=0$

(Answer : $3x^4-77x^3+720x^2-2876x+4058=0$)

(6) Show that the equation $x^4-3x^3+4x^2-2x+1=0$ can be transformed into a reciprocal equation by diminishing the roots by unity. Hence solve the equation.

(Answer : the roots are $\dfrac{\sqrt{5}+3\pm\sqrt{-10-2\sqrt{5}}}{4}, \dfrac{3-\sqrt{5}\pm\sqrt{2\sqrt{5}-10}}{4}$)

(7) Find the equation whose roots are the roots of the equation $x^4+8x^3+12x^2-16x-28=0$ each increased by 2. Hence solve the equation.

(Answer : $\pm\sqrt{2}, -4\pm\sqrt{2}$)

Summary

In this unit we have learned that the method of finding Sum of the powers of the roots of an equation, how transform a given equation. Also we have discussed the Reciprocal roots, finding roots of reciprocal equations and finally we studied Increase and decrease the roots of an equation by a quantity.

Further Reading

You can also refer the following books for further reading.

(1) Algebra by T.K.Manicavachagom Pillai and others Vol I

(2) Classical Algebra by Arumugam and others.

UNIT VIII

RELATIONS BETWEEN ROOTS AND COEFFICIENTS

Space for Hints

Unit Objectives

Unit Structure

Space for Hints

Objectives :

In this unit, we are going to discuss how to remove a particular term in the given equation, to find the nature of roots using Descarte's rule of sign, Rolle's theorem, to find multiple roots and solving cubic equations using Cardon's method.

After completing this unit, students may able to know

- o Removal of terms in an equation
- o Descarte's rule of sign
- o Rolle's theorem
- o Multiple roots
- o Cardon's method

8.1. Removal of terms in an equation

In this section we are going to study how to remove a particular term in an equation and using this we shall solve the equations.

Consider an equation $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \text{L} + a_n = 0$ ----- (8.1)

Substitute $y = x - h$ in (8.1), we get,

$$a_0(y-h)^n + a_1(y-h)^{n-1} + a_2(y-h)^{n-2} + \text{L} + a_n \quad \text{---------------- (8.2)}$$

After arranging the terms of (8.2), we have,

$$a_0y^n + (na_0h + a_1)y^{n-1} + \left\{\frac{n(n-1)}{2!}a_0h^2 + (n-1)a_1h + a_2\right\}y^{n-2} + \text{L} = 0 \quad \text{---------- (8.3)}$$

If we want to remove second term of (8.1), then put $na_0h + a_1 = 0$ in (8.3).

Similarly, if we want to remove third term of (8.1), then put $\frac{n(n-1)}{2!}a_0h^2 + (n-1)a_1h + a_2 = 0$ in (8.3).

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Example 8.1.1 :

Remove the second term from the equation $x^3 - 6x^2 + 10x - 3 = 0$

Proof : Given that $x^3 - 6x^2 + 10x - 3 = 0$ ------------------------------ (8.4)

Let us diminish the roots of (8.4) by h we have,

$(x+h)^3 - 6(x+h)^2 + 10(x+h) - 3 = 0$

(i.e) $x^3 + (3h-6)x^2 + \mathrm{L} = 0$ -- (8.5)

If we consider $3h - 6 = 0$ then $h = 2$

Thus to remove the second term of (8.4) decrease the roots of (8.4) by 2.

The operations of decreasing the roots by 2 are shown below :

2	1	-6	10	-3
	0	2	-8	4
	1	-4	2	1
	0	2	-4	
	1	-2	-2	
	0	2		
	1	0		

The transformed equation is $x^3 - 2x + 1 = 0$

Example 8.1.2 :

Remove the second term from the equation $x^5 + 5x^4 + 3x^3 + x^2 + x + 1 = 0$

Proof : Given that $x^5 + 5x^4 + 3x^3 + x^2 + x + 1 = 0$ ------------------- (8.6)

Let us diminish the roots of (8.6) by h we have,

$(x+h)^5 + 5(x+h)^4 + 3(x+h)^3 + (x+h)^2 + (x+h) + 1 = 0$

(i.e) $x^5 + (5h+5)x^4 + \mathrm{L} = 0$

If we consider $5h + 5 = 0$ then $h = -1$

Thus to remove the second term of (8.6) decrease the roots of (8.6) by 2.

The operations of decreasing the roots by 2 are shown below :

Space for Hints

-1	1	5	3	1	1	1
	0	-1	-4	1	-2	1
	1	4	-1	2	-1	2
	0	-1	-3	4	-6	
	1	3	-4	6	-7	
	0	-1	-2	6		
	1	2	-6	12		
	0	-1	-1			
	1	1	-7			
	0	-1				
	1	0				

The transformed equation is $x^5 - 7x^3 + 12x^2 - 7x + 2 = 0$

Example 8.1.3 :

Solve the equation $x^4 - 12x^3 + 48x^2 - 72x + 35 = 0$ by removing the second term.

Proof : Given that $x^4 - 12x^3 + 48x^2 - 72x + 35 = 0$ ----------------- (8.7)

Let us diminish the roots of (8.7) by h we have,

$(x+h)^5 + 5(x+h)^4 + 3(x+h)^3 + (x+h)^2 + (x+h) + 1 = 0$

(i.e) $x^5 + (5h+5)x^4 + L = 0$ -- (8.8)

If we consider $5h+5=0$ then $h=-1$

Thus to remove the second term of (8.7), it is enough to decrease the roots of (8.7) by 2.

The operations of decreasing the roots by 2 are shown below :

3	1	-12	48	-72	35
	0	3	-27	63	-27
	1	-9	21	-9	8
	0	3	-18	9	
	1	-6	3	0	
	0	3	-9		
	1	-3	-6		
	0	3			
	1	0			

The transformed equation is $x^4 - 6x^2 + 8 = 0$ -------------------------- (8.9)

Space for Hints

(i.e) $y^2 - 6y + 8 = 0$

(i.e) $(y-4)(y-2) = 0$

(i.e) $y - 4 = 0$ or $y - 2 = 0$

(i.e) $y = 4$ or $y = 2$

(i.e) $x^2 = 4$ or $x^2 = 2$

(i.e) $x = \pm 2$ or $x = \pm\sqrt{2}$

Thus the roots of (8.7) are $\pm 2 + 3$, $\pm\sqrt{2} + 3$

(i.e) $1, 5, 3 \pm \sqrt{2}$

Example 8.1.4 :

Solve the equation $x^4 + 16x^3 + 83x^2 + 152x + 84 = 0$ by removing the second term.

Proof : Given that $x^4 + 16x^3 + 83x^2 + 152x + 84 = 0$ ---------------- (8.10)

Let us diminish the roots of (8.7) by h we have,

$(x+h)^4 + 16(x+h)^3 + 83(x+h)^2 + 152(x+h) + 84 = 0$

(i.e) $x^4 + (4h+16)x^3 + \text{L} \;= 0$ -- (8.11)

If we consider $4h + 16 = 0$ then $h = -4$

Thus to remove the second term of (8.10), it is enough to decrease the roots of (8.10) by 2.

The operations of decreasing the roots by – 4 are shown below :

-4	1	16	83	152	84
	0	-4	-48	-140	-48
	1	12	35	12	36
	0	-4	-32	-12	
	1	8	3	0	
	0	-4	-16		
	1	4	-13		
	0	-4			
	1	0			

Space for Hints

The transformed equation is $x^4 - 13x^2 + 36 = 0$ ---------------------- (8.12) in which the second term is absent.

(i.e) $y^2 - 13y + 36 = 0$

(i.e) $(y-9)(y-4) = 0$

(i.e) $y - 9 = 0$ or $y - 4 = 0$

(i.e) $y = 9$ or $y = 4$

(i.e) $x^2 = 9$ or $x^2 = 4$

(i.e) $x = \pm 3$ or $x = \pm 2$

Thus the roots of (8.10) are $\pm 2 - 4, \pm 3 - 4$

(i.e) $-6, -2, -7, -1$

(i.e) $-7, -6, -2, -1$ are the required roots of the equation.

Example 8.1.5 :

Find the relation between the coefficients in the equation

$x^4 + px^3 + qx^2 + rx + s = 0$ in order that the coefficients of x^3 and x are removable by the same transformation.

Solution : Given that $x^4 + px^3 + qx^2 + rx + s = 0$ ---------------------- (8.13)

Let us diminish the roots of the equation (8.13) by h.

$\therefore$ put $x + h$ in the place of x in (8.13), we have,

$(x+h)^4 + p(x+h)^3 + q(x+h)^2 + r(x+h) + s = 0$

(i.e) $(x^4 + 4hx^3 + 6h^2x^2 + 4h^3x + h^4) + (x^3 + 3hx^2 + 3h^2x + h^3)$

$$+ q(x^2 + 2hx + h^2) + s = 0$$

(i.e) $x^4 + (4h+p)^3 + (6h^2 + 3ph + q)x^2 + (4h^3 + 3ph^2 + 2qh + r)x$

$+ (h^4 + ph^3 + qh^2 + rh + s) = 0$ ------------------ (8.14)

Now the coefficient of x^3 and x in (8.14) are zero then

$4h + p = 0$ -- (8.15)

$4h^3 + 3ph^2 + 2qh + r = 0$ -- (8.16)

From (8.15), $h = -\dfrac{p}{4}$ -- (8.17)

From (8.16) and (8.17), we get,

$$4\left(-\frac{p}{4}\right)^3+3p\left(-\frac{p}{4}\right)^2+2q\left(-\frac{p}{4}\right)+r=0$$

(i.e) $-\frac{1}{16}p^3+\frac{3}{16}p^3-\frac{1}{2}pq+r=0$

(i.e) $-p^3+3p^3-8pq+16r=0$

(i.e) $2p^3-8pq+16r=0$

(i.e) $p^3-4pq+8r=0$

which is the required condition.

Check your progress

Questions :

(1) Transform the equation $2x^3-9x^2+13x-6=0$ into one in which the second term is missing and hence solve the given equation.

(2) Solve the equation $x^4+16x^3+72x^2+64x-129=0$ by removing the second term.

(3) Solve the equation $x^4-8x^3+19x^2-12x+2=0$ by removing the second term.

8.2. Descarte's rule of sign

Definition : (Continuation of sign)

In any polynomial $f(x)$, whose terms are arranged in order, when a positive sign follows a positive sign or a negative sign followed by a negative sign then *continuation of sign* is said to occur.

Definition : (Variation of sign)

In any polynomial $f(x)$, whose terms are arranged in order, when a positive sign follows a negative sign or a negative sign followed by a positive sign then *variation of sign* is said to occur.

Space for Hints

Space for Hints

Definition : (*Ambiguity*)

When any term of a polynomial $f(x)$ has the double sign $\pm$ or m, an *ambiguity* is said to occur.

Definition : (complete equation)

An equation $f(x)=0$ is said to be complete if no coefficient is zero.

Note : In a complete equation, if r is the number of variations and s is the number of continuations the $r+s=n$ = degree of $f(x)$.

Result : If a polynomial $f(x)$ be multiplied by a factor $x-h$ $(h>0)$, then the resulting polynomial will have atleast one more changes of sign than in the original polynomial.

Proof : Let the sign of terms in the polynomial $f(x)$ be

$$+ \quad + \quad - \quad - \quad + \quad - \quad + \quad - \quad - \quad +$$

Here $f(x)$ have six changes of sign.

Now multiply $f(x)$ by $x-h$ $(h>0)$ whose signs are $+-$.

Consider the multiplication of $f(x)$ by $x-h$ and the product of signs are shown below :

	+	+	−	−	+	−	+	−	−	+
									+	−
	−	−	+	+	−	+	−	+	+	−
+	+	−	−	+	−	+	−	−	+	
+	m	−	±	+	−	+	−	±	+	−

Here $\pm$ or m indicates the ambiguities of sign, the term may be positive or negative or even zero.

Let us consider the worst case as by considering the ambiguities be replaced by continuation.

Then the sequence of signs in the product is

$$+ \quad + \quad - \quad - \quad + \quad - \quad + \quad - \quad - \quad +$$

This gives seven changes of sign .

(i.e) one more than the number of changes of sign in $f(x)$.

In other cases will, however, yield even more changes of sign then seven.

Thus, multiplication of polynomial by a factor of the form $x-h \quad (h>0)$ introduces at least one more additional changes of sign.

This proves the result.

Space for Hints

Descarte's rule of sign

Statement : (i) The number of positive roots cannot exceed the number of changes of sign in $f(x)$

(ii) The number of negative roots cannot exceed the number of changes of sign in $f(-x)$

Case (i) : For positive roots.

Let $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_r$ be the positive roots of $f(x)=0$.

Let $g(x)$ be the polynomial formed by the factors corresponding to negative and complex roots of $f(x)=0$.

Suppose $g(x)$ has m changes of sign where $m \geq 0$.

Then $(x-\alpha_1)g(x)$ has at least $m+1$ changes of sign,

$(x-\alpha_1)(x-\alpha_2)g(x)$ has at least $m+2$ changes of sign,

$(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)g(x)$ has at least $m+3$ changes of sign, and so on,

we have, $f(x)=(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_r)g(x)$ has at least $m+r$ changes of sign

Now $m \geq 0$

$\Rightarrow m+r \geq r$

$\Rightarrow$ number of positive roots of $f(x)=0$ cannot exceed the changes of sign in $f(x)$

Case (ii) : For negative roots.

We know that the negative roots of $f(x)=0$ are the positive roots of $f(-x)=0$.

Since positive roots of $f(x)=0$ cannot exceed the changes of sign in $f(-x)$.

$\therefore$ negative roots of $f(x)=0$ cannot exceed then changes of sign in $f(x)$

Space for Hints

This proves Descarte's rule of sign.

Note (1) : If there is only one change of sign in $f(x)$, the equation $f(x)=0$ has one and only one positive real root.

Note (2) : If there is only one change of sign in $f(-x)$, the equation $f(x)=0$ has one and only one negative real root.

Note (2) : If $f(x)=0$ is an equation of degree n with constant term $\neq 0$ and if r and s are number of variations of sign in $f(x)$ and $f(-x)$ respectively then $f(x)=0$ has at least $n-(r+s)$ complex roots.

Example 8.2.1 :

Apply Descarte's rule of sign, discuss the nature of roots of the equation $x^4+15x^2+7x-11=0$

Solution : Given that $x^4+15x^2+7x-11=0$ ------------------------ (8.18)

Let $f(x)=x^4+15x^2+7x-11$

The series of signs of the terms of the equation (8.18) are as follows :

$$+ \quad + \quad + \quad -$$

Here only one changes of sign and hence (8.18) has a positive root.

Now consider $f(-x)$

(i.e) $f(-x)=x^4+15x^2-7x-11$ -- (8.19)

Here the series of signs of the terms of the equation (8.19) are as follows :

$$+ \quad + \quad - \quad -$$

Here only one changes of sign and hence (8.19) has a positive root and hence (8.18) has one negative root.

Now $f(x)$ is of degree 4, therefore, (8.18) has 4 roots.

Since (8.18) has two real roots, the remaining two are imaginary roots.

Example 8.2.2 :

Show that the equation $x^{12}-15x^4-x^2+1=0$ has at least six complex roots

Solution : Given that $x^{12}-15x^4-x^2+1=0$ ------------------------- (8.20)

Let $f(x)=x^{12}-15x^4-x^2+1$

The series of signs of the terms of the equation (8.20) are as follows :

$$+ \quad - \quad - \quad +$$

There are two changes of sign for $f(x)$ and $\therefore$ $f(x)$ has at most two positive roots.

Now consider $f(-x)$

(i.e) $f(-x)=x^{12}-15x^4-x^2+1$ -- (8.21)

Here the series of signs of the terms of the equation (8.21) are as follows :

$$+ \quad + \quad - \quad -$$

Here two changes of sign for (8.21) and hence it has at most two negative roots

Thus (8.20) has at most 4 real roots.

Now $f(x)$ is of degree 12, therefore, (8.20) has at least six imaginary roots.

Example 8.2.3 :

Find the minimum number of imaginary roots of the equation $2x^7-x^4+4x^3-5=0$.

Solution : Given that $2x^7-x^4+4x^3-5=0$ ------------------------- (8.22)

Let $f(x)=2x^7-x^4+4x^3-5$

The series of signs of the terms of the equation (8.22) are as follows :

$$+ \quad - \quad + \quad -$$

There are three changes of sign for $f(x)$ and $\therefore$ $f(x)$ has at most three positive roots.

Now consider $f(-x)$

(i.e) $f(-x)=-2x^7-x^4-4x^3-5$ ------------------------------------- (8.23)

Space for Hints

Space for Hints

In $f(-x)$, there is no changes of sign and hence (8.23) has no positive roots

Thus (8.22) has no negative roots.

Hence $f(x)$ has at most three real roots.

Since degree of $f(x)$ is 7 and therefore $f(x)$ has 7 roots and hence $f(x)$ has $7-3=4$ imaginary roots.

Example 8.2.4 :

Prove that the equation $x^4+3x-1=0$ has two real and two imaginary roots.

Solution : Given that $x^4+3x-1=0$ ------------------------------------ (8.24)

Let $f(x)=x^4+3x-1=0$

The series of signs of the terms of the equation (8.24) are as follows :

$$+ \quad + \quad -$$

Since there is only one change of sign and degree of $f(x)$ is even, therefore $f(x)$ has one positive root.

Now consider $f(-x)$

(i.e) $f(-x)=x^4-3x-1=0$ --- (8.25)

Again $f(-x)$ has one change of sign and degree of $f(-x)$ is even, therefore $f(-x)$ has one positive root.

(i.e) $f(x)$ has one negative root.

Hence $f(x)$ has two real roots.

Since degree of $f(x)$ is 4 and therefore $f(x)$ has 4 roots and among two are real and the remaining two are imaginary roots.

8.3. Rolle's Theorem

Statement : Between two consecutive roots a and b of the equation $f(x)=0$ where $f(x)$ is a polynomial, there lies at least one real root of the equation $f'(x)=0$.

Proof : Let a and b be two consecutive roots of the equation $f(x)=0$.

Let $f(x)=(x-a)^m(x-b)^n g(x)$ where $m,n\in\mathbf{N}$ and $g(a)\neq 0$ and $g(b)\neq 0$

Since a and b be two consecutive roots of the equation $f(x)=0$, then the sign of $g(x)$ in the interval $a\le x\le b$ is either positive throughout or negative throughout, because, if $g(a)$ and $g(b)$ are of different signs then there is real root of $g(x)=0$ that is of $f(x)=0$ lying between $a\le x\le b$, which is contrary to the hypothesis that a and b are consecutive roots.

Now $f'(x) = (x-a)^m n(x-b)^{n-1}g(x) + m(x-a)^{m-1}(x-b)^n g(x)$

$$+ (x-a)^m(x-b)^n g'(x)$$

(i.e) $f'(x) = (x-a)^{m-1}(x-b)^{n-1}h(x)$

where $h(x) = \{m(x-b)+n(x-a)\}g(x)+(x-a)(x-b)g'(x)$

$\therefore\ h(a) = m(a-b)g(a)$ and $h(b) = n(b-a)g(b)$

Now $h(a)$ and $h(b)$ have different signs since $g(a)$ and $g(b)$ have the same sign.

$\therefore\ g(x)=0$ has at least one root between a and b.

Hence $f'(x)=0$ has at least one root between a and b.

Note (1) : If all the roots of $f(x)=0$ are real, then all the roots of $f'(x)=0$ are also real.

Note (2) : If $f(x)=0$ is a polynomial of degree n, then $f'(x)=0$ is a polynomial of degree $n-1$ and each root of $f'(x)=0$ lies in each of the $n-1$ intervals between the n roots of $f(x)=0$.

Note (3) : At the most only one real root of $f(x)=0$ can lie between two consecutive roots of $f'(x)=0$, that is the real roots of $f'(x)=0$ separate those of $f(x)=0$.

Note (4) : If $f'(x)=0$ has r real roots, then $f(x)=0$ cannot have more than $r+1$ roots.

Note (5) : $f(x)=0$ has at least as many imaginary roots as $f'(x)=0$

Space for Hints

Space for Hints

Example 8.3.1 :

Find the nature of roots of $2x^3 - 9x^2 + 12x + 3 = 0$

Solution : Let $f(x) = 2x^3 - 9x^2 + 12x + 3 = 0$

$\therefore \; f'(x) = 6x^2 - 18x + 12$

$= 6(x^2 - 3x + 2)$

$= 6(x-1)(x-2)$

Now the roots of $f'(x) = 0$ are 1 and 2.

Thus the roots $f(x) = 0$ will lie in the intervals $(-\infty, 1)$, $(1, 2)$ and $(2, \infty)$

Now we have the following table of signs of $f(x)$ at the points of the interval

x	$-\infty$	1	2	∞
$f(x)$	$-$	$+$	$+$	$+$

From the above table it is clear that $f(x) = 0$ has only one changes of sign

$\therefore \; f(x) = 0$ has one real root say α in the interval $(-\infty, 1)$

Now $f(0) = 3$ and $f(1) = 8$

(i.e) there is no changes of sign for $f(x) = 0$ between 0 and 1.

Thus $\alpha \notin (0,1)$

Hence $\alpha \in (-\infty, 1)$ and it implies that α is a negative root of $f(x) = 0$

Again degree of $f(x)$ is three and the remaining two roots are imaginary roots.

Hence $f(x) = 0$ has one negative root and two imaginary roots.

Example 8.3.2 :

Find the nature of roots of $x^4 + 4x^3 - 20x^2 + 10 = 0$

Solution : Let $f(x) = x^4 + 4x^3 - 20x^2 + 10 = 0$

$\therefore \; f'(x) = 4x^3 + 12x^2 - 40x = 0$

$= 4x(x^2 + 3x - 10)$

$= 4x(x+5)(x-2)$

Now the roots of $f'(x) = 0$ are -5, 0 and 2.

$\therefore\ f'(x)=0$ has 3 real roots

$\Rightarrow\ f(x)=0$ cannot have more than 4 real roots.

Thus the roots $f(x)=0$ will lie in the intervals $(-\infty,-5)$, $(-5,0)$, $(0,2)$ and $(2,\infty)$

Now we have the following table of signs of $f(x)$ at the points of the interval

x	$-\infty$	-5	0	2	∞
$f(x)$	+	–	+	–	+

From the above table it is clear that $f(x)=0$ has four changes of sign

$\therefore\ f(x)=0$ has four real roots because degree of $f(x)$ is 4 and all the four real roots of $f(x)=0$ will lie in the four intervals $(-\infty,-5)$, $(-5,0)$, $(0,2)$ and $(2,\infty)$

Hence $f(x)=0$ has two negative root and two positive roots.

Example 8.3.3 :

Find the nature of roots of $3x^4-8x^3-6x^2+24x-7=0$

Solution : Let $f(x)=3x^4-8x^3-6x^2+24x-7=0$

$$\begin{aligned}\therefore\ f'(x) &= 12x^3-24x^2-12x+24\\ &= 12(x^3-2x^2-x+2)\\ &= 12(x+1)(x-1)(x-2)\end{aligned}$$

Now the roots of $f'(x)=0$ are -1, 1 and 2.

$\therefore\ f'(x)=0$ has 3 real roots

$\Rightarrow\ f(x)=0$ cannot have more than 4 real roots.

Thus the roots $f(x)=0$ will lie in the intervals $(-\infty,-1)$, $(-1,1)$, $(1,2)$ and $(2,\infty)$

Now we have the following table of signs of $f(x)$ at the points of the interval

x	$-\infty$	-1	1	2	∞
$f(x)$	+	–	+	+	+

From the above table it is clear that $f(x)=0$ has two changes of sign

Space for Hints

Space for Hints

Thus $f(x)=0$ has at most two real roots.

Clearly one real root lies in the interval $(-\infty,-1)$ and $\therefore$ it is a negative root.

Again $f(0)=-7$ and hence, the second real root lies in the interval $(0,1)$ and thus the second root is positive.

$\therefore$ $f(x)=0$ has four roots because degree of $f(x)$ is 4 and hence $f(x)=0$ has one negative root, one positive roots and two imaginary roots.

Example 8.3.4 :

Find the range of k for which $x^4+4x^3-2x^2-12x+k=0$ are real roots.

Solution : Let $f(x)=x^4+4x^3-2x^2-12x+k$

$\therefore$ $f'(x)=4x^3+12x^3-4x^2-12$

$$= 4(x^3+3x^3-x^2-3)$$

$$= 4(x+3)(x+1)(x-1)$$

Thus $f'(x)=0$ if $x=-3,-1,1$ which are real.

Thus the roots $f(x)=0$ will lie in the intervals $(-\infty,-3)$, $(-3,-1)$, $(-1,1)$ and $(1,\infty)$

Now we have the following table of signs of $f(x)$ at the points of the interval

x	$-\infty$	-3	-1	1	∞
$f(x)$	$+$	$k-9$	$k+7$	$k-9$	$+$

Given that $f(x)=0$ has all real roots.

This is possible only when $k-9<0$ -------------------------------------- (8.26)

$k+7>0$ -------------------------------------- (8.27)

$k-9<0$ -------------------------------------- (8.28)

From (8.26), $k<9$ -- (8.29)

From (8.27), $k>-7$ -- (8.30)

From (8.28), $k<9$ -- (8.31)

From (8.30) and (8.31), we have $-7<k<9$

Thus, when $-7<k<9$, then $f(x)=0$ has all roots are real.

Example 8.3.5 :

Find the range of k for which $3x^4-4x^3-12x^2+k=0$ are real roots.

Solution : Let $f(x)=3x^4-4x^3-12x^2+k=0$

$\therefore\ f'(x)=12x^3-12x^2-24x$

$$=12x(x^2-x-2)$$

$$=12x(x+1)(x-2)$$

Thus $f'(x)=0$ if $x=-1,\ 0,\ 2$ which are real.

Thus the roots $f(x)=0$ will lie in the intervals $(-\infty,-1)$, $(-1,0)$, $(0,2)$ and $(2,\infty)$

Now we have the following table of signs of $f(x)$ at the points of the interval

x	$-\infty$	-1	0	2	∞
$f(x)$	$+$	$k-5$	k	$k-32$	$+$

Given that $f(x)=0$ has all real roots.

This is possible only when $k-5<0$ ------------------------------------ (8.32)

$k>0$ -- (8.33)

$k-32<0$ ---------------------------------- (8.34)

From (8.32), $k<5$ -- (8.35)

From (8.33), $k>0$ -- (8.36)

From (8.34), $k<32$ -- (8.37)

From (8.35), (8.36) and (8.37), we have $0<k<5$

Thus, when $0<k<5$, then $f(x)=0$ has all roots are real.

Check your progress

Question

(1) Find the nature of roots of the equation $3x^4-8x^3-6x^2+24x-7=0$

(2) Find the nature of roots of the equation $4x^3-21x^2+18x+20=0$

(3) Find the range of k for which $3x^4+8x^3-6x^2-24x+k=0$ are real roots.

((–13, –8))

Space for Hints

Space for Hints

(4) Find the range of k for which $2x^3-9x^2+12x-k=0$ are real roots.

((4, 5))

(answers :

(1) – one negative root, one positive root and two imaginary roots

(2) – three real roots

(3) $-13<k<-8$

(4) $4<k<5$)

8.4. Multiple Roots

If $f(x)$ is a polynomial in x and the equation $f(x)=0$ has m equal roots equal to α, then $f(x)$ must be of the form $(x-\alpha)^m g(x)$ where $g(\alpha)\neq 0$.

(i.e) $f(x)=(x-\alpha)^m g(x)$

Now $f'(x)=(x-\alpha)^m g'(x)+m(x-\alpha)^{m-1} g(x)$

$= (x-\alpha)^{m-1}((x-\alpha)g'(x)+mg(x))$

Hence $(x-\alpha)^{m-1}$ is a common factor of $f(x)$ and $f'(x)$

Again $(x-\alpha)^{m-1}$ will not be a common factor unless $f(x)$ is divisible by $(x-\alpha)^m$.

Procedure for finding multiple roots of an equation $f(x)=0$

Step 1 : Find $f'(x)$

Step 2 : Find the **H.C.F.** of $f(x)$ and $f'(x)$

Step 3 : Find the roots of H.C.F.

Each different root of the H.C.F will occur once more in $f(x)$ than it does in the H.C.F.

Example 8.4.1 :

Find the multiple roots of the equation $4x^3+20x^2-23+6=0$

Space for Hints

Solution : Given that $4x^3+20x^2-23+6=0$

Let $f(x)=4x^3+20x^2-23+6$

$\therefore\ f'(x)=12x^2+40x-23$

Now we shall find the H.C.F. of $f(x)$ and $f'(x)$

6	12	40	-23	4	20	-23	6	
	12	-6					x 3	
23		46	-23	12	60	-69	18	1
		46	-23	12	40	-23		
			0		20	-46	18	
				x 6	120	-276	108	10
					120	400	-230	
						-676	338	
					÷ (-338)	2	-1	

From the above table it is clear that $2x-1$ is common factor of $f(x)$ and $f'(x)$

Thus $(2x-1)^2$ is a factor of $f(x)$

(i.e) $\frac{1}{2}$ is a double root of $f(x)$

Let α be the third root of $f(x)=0$

Thus the product of the roots of $f(x)=0$ is $-\frac{6}{4}$

(i.e) $\frac{1}{2}\cdot\frac{1}{2}\cdot\alpha=-\frac{6}{4}$

(i.e) $\alpha=-6$

Hence the roots of the equation are $\frac{1}{2},\frac{1}{2},-6$.

Example 8.4.2 :

Find the multiple roots of the equation $x^4+7x^3+17x^2+17x+6=0$

Solution : Given that $x^4+7x^3+17x^2+17x+6=0$

Let $f(x)=x^4+7x^3+17x^2+17x+6$

$\therefore\ f'(x)=4x^3+21x^2+34x+17$

Space for Hints

Now we shall find the H.C.F. of $f(x)$ and $f'(x)$

	4	21	34	17	1	7	17	17	6	
				×7					×4	
	28	147	238	119	4	28	68	68	24	
				÷ 7	4	21	34	17		
	4	21	34	17		7	34	51	24	
				×11					×4	
1	44	231	374	187		28	136	204	96	
	44	136	92			28	147	238	119	
		95	282	187			–11	–34	–23	
				×11					×(–1)	
		1045	3102	2057			11	34	23	
1		1045	3230	2185					×4	
			–128	-128			44	136	92	
				÷128					÷4	
			1	1			11	34	23	11
							11	11		
								23	23	23
								23	23	
									0	

From the above table it is clear that $x+1$ is common factor of $f(x)$ and $f'(x)$

Thus $(x+1)^2$ is a factor of $f(x)$

(i.e) -1 is a double root of $f(x)$

Now

$$
\begin{array}{r|l}
 & x^2+5x+6 \\
\hline
x^2+5x+6 & x^4+7x^3+17x^2+17x+6 \\
 & x^4+2x^3+6x^2 \\
\hline
 & \quad 5x^3+16x^2+17x \\
 & \quad 5x^3+10x^2+5x \\
\hline
 & \qquad 6x^2+12x+6 \\
 & \qquad 6x^2+12x+6 \\
\hline
 & \qquad\quad 0
\end{array}
$$

Space for Hints

Thus $f(x) = (x+1)^2(x^2+5x+6)$

$\therefore\ f(x)=0 \Rightarrow (x+1)^2(x^2+5x+6)=0$

(i.e) $(x+1)^2=0$ or $x^2+5x+6=0$

(i.e) $x=-1$ or $x^2+5x+6=0$

Now $x^2+5x+6=0$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow x = \frac{-5 \pm \sqrt{25-24}}{2}$$

$$\Rightarrow x = \frac{-5 \pm 1}{2}$$

$$\Rightarrow x = \frac{-5-1}{2} \text{ or } x = \frac{-5+1}{2}$$

$\Rightarrow x=-3$ or $x=-2$

Hence the roots of the equation are $-1, -1, -2, -3$.

Example 8.4.3 :

Find the multiple roots of the equation $4x^4+24x^3+49x^2+45x+25=0$

Solution : Given that $4x^4+24x^3+49x^2+45x+25=0$

Let $f(x) = 4x^4+24x^3+49x^2+45x+25$

$\therefore\ f'(x) = 16x^3+72x^2+98x+45$

Now we shall find the H.C.F. of $f(x)$ and $f'(x)$

Space for Hints

	16	72	98	45	4	24	49	45	25	$\frac{1}{4}$
	16	$\frac{96}{5}$	- 52		4	18	$\frac{49}{2}$	$\frac{45}{4}$		
$-\frac{528}{25}$		$\frac{264}{5}$	150	45		6	$\frac{49}{2}$	$\frac{135}{4}$	25	$\frac{3}{8}$
		$\frac{264}{5}$	$\frac{1584}{25}$	$-\frac{858}{5}$		6	27	$\frac{147}{4}$	$\frac{135}{8}$	
			$\frac{2166}{25}$	$\frac{1083}{5}$			$-\frac{5}{2}$	-3	$\frac{65}{8}$	$-\frac{125}{4332}$
				$\times\frac{25}{1083}$			$-\frac{5}{2}$	-25		
			2	5				$\frac{13}{4}$	$\frac{65}{8}$	$\frac{325}{8664}$
								$\frac{13}{4}$	$\frac{65}{8}$	
								0		

From the above table it is clear that $2x+5$ is common factor of $f(x)$ and $f'(x)$

Thus $(2x+5)^2$ is a factor of $f(x)$

(i.e) $-\frac{5}{2}$ is a double root of $f(x)$

Now $f(x) = (2x+5)^2(x^2+x+1)$

$\therefore\ f(x)=0 \Rightarrow (2x+5)^2(x^2+x+1)=0$

(i.e) $(2x+5)^2=0$ or $x^2+x+1=0$

(i.e) $x=-\frac{5}{2}$ or $x^2+x+1=0$

Now $x^2+x+1=0$

$$\Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow x=\frac{-1\pm\sqrt{1-4}}{2}$$

$$\Rightarrow x = \frac{-1 \pm i\sqrt{3}}{2}.$$

Hence the roots of the equation are $-\frac{5}{2}, -\frac{5}{2}, \frac{-1 \pm i\sqrt{3}}{2}$.

Example 8.4.4 :

Find the multiple roots of the equation $x^4 - 6x^3 + 13x^2 - 24x + 36 = 0$

Solution : Given that $x^4 - 6x^3 + 13x^2 - 24x + 36 = 0$

Let $f(x) = x^4 - 6x^3 + 13x^2 - 24x + 36$

$\therefore f'(x) = 4x^3 - 18x^2 + 26x - 24$

Now we shall find the H.C.F. of $f(x)$ and $f'(x)$

	2	−9	13	−12	1	−6	13	−24	36	
−2	2	66	−216						×2	
		−75	229	−12	2	−12	26	−48	72	1
		−75	−2475	8100	2	−9	13	−12		
			2704	−8112		−3	13	−36	72	
				÷2704					×2	
			1	−3		−6	26	−72	144	3
						−6	27	−39	36	
							−1	−33	108	−1
							−1	3		
								−36	108	
								−36	108	
								0		

From the above table it is clear that $x-3$ is common factor of $f(x)$ and $f'(x)$

Thus $(x-3)^2$ is a factor of $f(x)$

(i.e) 3 is a double root of $f(x)$

Now $f(x) = (x-3)^2(x^2+4)$

$\therefore f(x) = 0 \Rightarrow (x-3)^2(x^2+4) = 0$

(i.e) $(x-3)^2 = 0$ or $x^2 + 4 = 0$

(i.e) $x = 3$ or $x^2 = -4$

(i.e) $x = 3$ or $x = \pm 2i$

Hence the roots of the equation are $3, 3, \pm 2$.

Space for Hints

Space for Hints

Example 8.4.5 :

Find the value of k for which $x^3+4x^2+5x+2+k=0$ has equal roots

Solution : Given that $x^3+4x^2+5x+2+k=0$

Let $f(x)=x^3+4x^2+5x+2+k$

$\therefore\ f'(x)=3x^2+8x+5$

Now we shall find the H.C.F. of $f(x)$ and $f'(x)$

$-\frac{27}{2}$	$3x^2+8x+5$	$x^3+4x^2+5x+2+k$	$\frac{1}{3}x$
	$3x^2+3x-\frac{27}{2}kx$	$x^3+\frac{8}{3}x^2+\frac{5}{3}x$	
	$5x+5+\frac{27}{2}kx$	$\frac{4}{3}x^2+\frac{10}{3}x+2+k$	$\frac{4}{9}$
		$\frac{4}{3}x^2+\frac{32}{9}x+\frac{20}{9}$	
		$-\frac{2}{9}x-\frac{2}{9}+k$	

If $f(x)$ and $f'(x)$ have a linear common factor then

$$\frac{5+\frac{27}{2}k}{-\frac{2}{9}}=\frac{5}{-\frac{2}{9}+k}$$

(i.e) $\dfrac{\frac{10+27k}{2}}{-\frac{2}{\cancel{9}}}=\dfrac{5}{\frac{-2+k}{\cancel{9}}}$

(i.e) $\dfrac{10+27k}{-4}=\dfrac{5}{-2+k}$

(i.e) $(10+37k)(-2+k)=-20$

(i.e) $-20+36k+243k^2=-20$

(i.e) $k(36+243k)=0$

(i.e) $k=0$ or $36+243k=0$

(i.e) $k=0$ or $243k=-36$

(i.e) $k=0$ or $k=-\frac{4}{27}$

When $k=0$ then the linear factor on both sides becomes $5x+5$ and $-\frac{2}{9}x-\frac{2}{9}$

$\therefore$ the H.C.F. is $x+1$

Hence $(x+1)^2$ is factor of $f(x)$

(i.e) $x=-1$ is a double root of $f(x)$.

If α is the third root of $f(x)$, then product of the roots of $f(x)=0$ is

$(-1)(-1)\alpha=-2$

(i.e) $\alpha=-2$

Thus, in this case the roots are $-1, -1, -2$

When $k=-\frac{4}{27}$ then the linear factor on both sides becomes $3x+5$ and $\frac{-6x-10}{27}$

$\therefore$ the H.C.F. is $3x+5$

Hence $(3x+5)^2$ is factor of $f(x)$

(i.e) $x=-\frac{5}{3}$ is a double root of $f(x)$.

If β is the third root of $f(x)$, then product of the roots of $f(x)=0$ is

$$\left(-\frac{5}{3}\right)\left(-\frac{5}{3}\right)\beta=-\left(2-\frac{4}{27}\right)$$

(i.e) $\left(\frac{25}{9}\right)\beta=-\left(\frac{54-4}{27}\right)$

(i.e) $\left(\frac{25}{9}\right)\beta=-\left(\frac{50}{27}\right)$

(i.e) $\beta=-\frac{2}{3}$

Thus, in this case the roots are $-\frac{5}{3}, -\frac{5}{3}, -\frac{2}{3}$.

Example 8.4.6 :

Prove that the equation $x^3-3qx+2r=0$ has a double root if $q^3=r^2$

Space for Hints

Space for Hints

Solution : Given that $x^3-3qx+2r=0$

Let $f(x)=x^3-3qx+2r$

$\therefore\ f'(x)=3x^2-3q$

Let α be a double root of $f(x)=0$

Thus α is a common root of $f(x)=0$ and $f'(x)=0$

(i.e) $f(\alpha)=0$ and $f'(\alpha)=0$

(i.e) $\alpha^3-3q\alpha+2r=0$ -- (8.38)

and $f'(x)=0$

(i.e) $3\alpha^2-3q=0$ -- (8.39)

Now (8.38) $-\alpha\times$ (8.39) $\Rightarrow -2q\alpha+2r=0$

(i.e) $\alpha=\dfrac{r}{q}$

Thus (8.39) $\Rightarrow 3\left(\dfrac{r}{q}\right)^2-3q=0$

$\Rightarrow \dfrac{3r^2}{q^2}-3q=0$

$\Rightarrow 3r^2-3q^3=0$

$\Rightarrow r^2-q^3=0$

$\Rightarrow q^3=r^2$

This proves the problem.

Example 8.4.7 :

Prove that the equation $x^5-5x^3+5x^2-1=0$ has three roots equal and find the roots.

Solution : Given that $x^5-5x^3+5x^2-1=0$

Let $f(x)=x^5-5x^3+5x^2-1$

$\therefore\ f'(x)=5x^4-15x^2+10x$

and $f''(x)=20x^3-30x+10$

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and $f'''(x) = 60x^2 - 30$

Now $f''(x) = 20x^3 - 30x + 10$

$$= 5(4x^3 - 6x + 2)$$

$$= 5(x-1)(2x^2 + 2x - 1)$$

Clearly $f''(1) = 0$

Now $f(1) = 0$, $f'(x) = 0$, $f''(1) = 0$ and $f'''(x) \neq 0$

$\therefore$ $x = 1$ is root of multiplicity 3 for $f(x) = 0$

Now we shall find the roots of $f(x) = 0$ by using synthetic division .

1	1	0	-5	5	0	-1
	0	1	1	-4	1	1
1	1	1	-4	1	1	0
	0	1	2	-2	-1	
1	1	2	-2	-1	0	
	0	1	3	1		
	1	3	1	0		

Now $f(x) = (x-1)^3(x^2 + 3x + 1)$

Thus $f(x) = 0$

$$\Rightarrow (x-1)^3(x^2 + 3x + 1) = 0$$

$$\Rightarrow (x-1)^3 = 0 \text{ or } x^2 + 3x + 1 = 0$$

$$\Rightarrow x = 1,1,1 \text{ or } x^2 + 3x + 1 = 0$$

Now $x^2 + 3x + 1 = 0$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{-3 \pm \sqrt{9-4}}{2}$$

$$\Rightarrow x = \frac{-3 \pm \sqrt{5}}{2}$$

Space for Hints

Thus the roots of the equation are $1,1,1,\frac{-3\pm\sqrt{5}}{2}$.

Check your progress

Question

(1) Find the multiple roots of the equation $8x^3-20x^2+6x+9=0$

(2) Find the multiple roots of the equation $x^5-x^4-4x^2+7x-3=0$

(3) Show that $x^3+qx+r=0$ has two equal roots if $27r^2+4q^3=0$

(4) Verify that the equation $4x^3-27x+27=0$ has two equal roots, if it so, find all the roots.

(Answer :

(1) $\frac{3}{2},\frac{3}{2},-\frac{1}{2}$

(2) 1, 1, 1, $-1\pm i\sqrt{2}$

(4) $\frac{3}{2},\frac{3}{2},-3$)

8.5. Strum's Theorem

Let $f(x)=0$ be an equation having no equal roots.

Let $f_1(x)=f'(x)$

Let the process of finding the greatest common measure of $f(x)$ and $f_1(x)$ be performed.

Let q_1 be the quotient and $f_2(x)$ the remainder with the sign changed.

Then $f(x)=q_1 f_1(x)-f_2(x)$

Similar operation can be adopted between $f_1(x)$ and $f_2(x)$, we get,

$f_1(x)=q_2 f_2(x)-f_3(x)$

Continuing like above, we have,

$f_2(x)=q_3 f_3(x)-f_4(x)$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$f_{r-1}(x) = q_r f_r(x) - f_{r+1}(x)$ and so on.

Clearly the functions $f_1(x), f_2(x), f_3(x), \text{L}$ will go on diminishing in degree until we reach a numerical remainder, say, $f_n(x)$.

These functions $f_1(x), f_2(x), f_3(x), \text{L}, f_n(x)$ are called ***Strum's functions.***

Now the difference between the number of changes of sign in the series of Strum's functions when a is substituted for x and the number when b is substituted for x express exactly the number of real roots of the equation $f(x) = 0$ between a and b.

Note (1) : Substitute 0 and ∞ in the series of Strum's functions and the difference between the number of changes of sign will give the positive roots.

Note (2) : Substitute $-\infty$ and 0 in the series of Strum's functions and the difference between the number of changes of sign will give the negative roots.

Note (3) : The difference between the number of changes of sign when $-\infty$ and $+\infty$ are substituted inn the series of Strum's function will give the number of real roots of the equation.

Example 8.5.1 :

Find the number of real roots of the equation $x^4 + 4x^3 - 4x - 13 = 0$

Solution : Given that $x^4 + 4x^3 - 4x - 13 = 0$

Let $f(x) = x^4 + 4x^3 - 4x - 13$

Thus $f_1(x) = f'(x) = 4x^3 + 12x^2 - 4$

(i.e) $f_1(x) = 4(x^3 + 3x^2 - 1)$

First we shall find, the Strum's functions for $f(x)$.

Space for Hints

Space for Hints

x	x^3+3x^2+0x-1 x^3+x^2+4x	$x^4+4x^3+0x^2-4x-13$ $x^4+3x^3+0x^2-x$	x
	$2x^2-4x-1$ $2x^2+2x+8$	$x^3+0x^2-3x-13$ x^3+3x^2+0x-1	1
	$-6x-9$ $\div(-3)$	$-3x^2-3x-12$ $\div(-3)$	
	$2x+3$	$f_2(x) = x^2+x+4$ $\times 2$	
		$2x^2+2x+8$ $2x^2+6x$	2
		$-4x+8$ $-4x-6$	-2
		14	
		$f_3(x) = -14$	

Thus the Strum's function of $f(x)$ are given by :

$$f(x)=x^4+4x^3-4x-13,$$

$$f_1(x)=4(x^3+3x-1),$$

$$f_2(x)=x^2+x+4,$$

$$f_3(x)=2x+3,$$

$$f_4(x)=-14$$

The following table shows the signs of Strum's functions by substituting $-\infty$, 0 and ∞ for x.

I	II	III	IV
x	$-\infty$	0	∞
$f(x)$	+	−	+
$f_1(x)$	−	−	+
$f_2(x)$	+	+	+
$f_3(x)$	−	+	+
$f_4(x)$	−	−	−
No. of changes of sign	3	2	1

Thus the number of real roots = $3-2=1$ (difference of changes of signs when $x=-\infty$ and $x=\infty$ in the Strum's functions)

Number of negative roots = $2-1=1$, and

Number of positive roots = $2-1=1$

Hence the given equation $f(x)=0$ has two real roots in which one is negative root and another is positive root.

Example 8.5.2 :

Find the number of real roots of the equation

$$x^5-5x^4+9x^3-9x^2+5x-1=0$$

Solution : Given that $x^5-5x^4+9x^3-9x^2+5x-1=0$

Let $f(x)= x^5-5x^4+9x^3-9x^2+5x-1$

Thus $f_1(x) = f'(x) = 5x^4-20x^3+27x^2-18x+5$

(i.e) $f_1(x) = 4(x^3+3x^2-1)$

First we shall find, the Strum's functions for $f(x)$.

$5x$	$5x^4-20x^3+27x^2-18x+5$ $5x^4+0x^3-5x^2$	$x^5-5x^4+9x^3-9x^2+5x-1$ $\times 5$	
$-2x$	$-20x^3+32x^2-18x+5$ $-20x^3+0x^2+20x$	$x^5-25x^4+45x^3-45x^2+25x-5$ $5x^5-20x^4+27x^3-18x^2+5x$	x
	$32x^2-38x+5$ $\div(-1)$	$-5x^4+18x^3-27x^2+20x-5$ $-5x^4+20x^3-27x^2+18x-5$	-1
	$f_3(x) = -32x^2+38x-5$ $\times 26$	$-2x^3+0x^2+2x$ $\div(-2)$	
	$-832x^2+988x-130$ $-832x^2+608x$	$f_2(x) = x^3+0x^2-x$ $\times 32$	
	$380x-130$ $\div 10$	$32x^3+0x^2+32x$ $32x^3-38x^2+5x$	$-x$
	$38x-13$ $\times 26$	$38x^2-37x$ $\times 32$	
	$988x-388$ $988x-722$	$1216x^2-1184x$ $1216x^2-1444x+190$	38
	384 $f_5(x) = -384$	$260x-190$ $\div(-10)$ $f_4(x) = 26x--19$	

Space for Hints

Space for Hints

Thus the Strum's function of $f(x)$ are given by :

$$f(x)=x^5-5x^4+9x^3-9x^2+5x-1,$$

$$f_1(x)=5x^4-20x^3+27x^2-18x+5,$$

$$f_2(x)=x^3-x,$$

$$f_3(x)=-32x^2+38x-5,$$

$$f_4(x)=26x-19,$$

$$f_5(x)=-384$$

The following table shows the signs of Strum's functions by substituting $-\infty$, 0 and ∞ for x.

I	II	III	IV
x	$-\infty$	0	∞
$f(x)$	$-$	$-$	$+$
$f_1(x)$	$+$	$+$	$+$
$f_2(x)$	$-$	$+$	$+$
$f_3(x)$	$-$	$-$	$-$
$f_4(x)$	$-$	$+$	$-$
$f_5(x)$	$-$	$-$	$-$
No. of changes of sign	2	4	1

Thus the number of real roots $= 4-1=3$ (difference of changes of signs when $x=-\infty$ and $x=\infty$ in the Strum's functions)

Number of positive roots $= 4-1=3$

Hence the given equation $f(x)=0$ has three real roots and two imaginary roots.

Example 8.5.3 :

Find the number and position of real roots of the equation $x^3-3x+1=0$.

Solution : Given that $f(x)=x^3-3x+1=0$

Let $f(x)=x^3-3x+1$

Thus $f_1(x) = f'(x) = 3x^2-3 = 3(x^2-1)$

First we shall find, the Strum's functions for $f(x)$.

	x^2+0x-1 $\times 2$	x^3+0x^2-3x+1 x^3+0x^2-x	x
1	$2x^2+0x-2$ $2x-1$	$-2x+1$	
		$f_2(x) = 2x-1$	
	$-2x-1$		
-1	$-2x+1$		
	-2		
	$f_3(x)=2$		

Thus the Strum's function of $f(x)$ are given by :

$f(x)=x^3-3x+1$,

$f_1(x)=3x^2-3$,

$f_2(x)=2x-1$,

$f_3(x)=2$,

The following table shows the signs of Strum's functions by substituting $-\infty$, 0 and ∞ for x.

I	II	III	IV	V	VI	VII	VIII
x	$-\infty$	0	∞	1	-1	2	-2
$f(x)$	$-$	$+$	$+$	$-$	$-$	$-$	$+$
$f_1(x)$	$+$	$-$	$+$	0	0	$+$	$+$
$f_2(x)$	$-$	$-$	$+$	$-$	$+$	$-$	$+$
$f_3(x)$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
No. of changes of sign	3	2	0	1	1	3	0

Thus the number of real roots $= 3-0=3$ (difference of changes of signs when $x=-\infty$ and $x=\infty$ in the Strum's functions)

Number of negative roots $= 3-2=1$

Number of positive roots $= 2-0=2$

Hence the given equation $f(x)=0$ has three real roots.

Space for Hints

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From the above table positive roots lies in the interval $[1,2)$ and negative root lies in the interval $[-1, 0)$.

Example 8.5.4 :

Find the number and position of real roots of the equation

$$x^4-3x^3-2x^2+7x+3=0$$

Solution : Given that $x^4-3x^3-2x^2+7x+3=0$

Let $f(x)= x^4-3x^3-2x^2+7x+3$

Thus $f_1(x) = f'(x) = 4x^3-9x^2-4x+7$

First we shall find, the Strum's functions for $f(x)$.

	$4x^3-9x^2-4x+7$ $\times 43$	$x^4-3x^3-2x^2+7x+3$ $\times 4$	
4	$172x^3-387x^2-172x+301$ $172x^3-288x^2-276x$	$4x^4-12x^3-8x^2+28x+12$ $4x^4-9x^3-4x^2+7x$	x
	$-99x^2+104x+301$ $\times 43$	$-3x^3-4x^2+21x+12$ $\times 4$	
-99	$-4257x^2+4472x+12943$ $-4257x^2+7128x+6831$	$-12x^3-16x^2+84x+48$ $-12x^3+27x^2+12x-21$	-3
	$-4257x^2 4472x+12943$	$-43x^2+72x+69$	
	$-2656x+6112$ $\div(-32)$	$f_2(x) = 43x^2-72x-69$ $\times 83$	
	$f_3(x)= 83x-191$	$3569x^2-5976x-5727$ $3569x^2-8213x$	43
		$2237x-5727$ $\times 83$	
		$185671x-475341$ $185671x-427267$	2237
		-48074	
		$f_4(x) = 48074$	

Thus the Strum's function of $f(x)$ are given by :

$$f(x)=x^4-3x^3-2x^2+7x+3,$$

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$f_1(x) = 4x^3 - 9x^2 - 4x + 7$,

$f_2(x) = 43x^2 - 72x - 69$,

$f_3(x) = 83x - 191$,

$f_4(x) = 48074$,

The following table shows the signs of Strum's functions by substituting $-\infty$, 0 and ∞ for x.

I	II	III	IV	V	VI	VII	VIII	IX
x	$-\infty$	0	∞	1	-1	2	-2	3
$f(x)$	+	+	+	+	–	+	+	+
$f_1(x)$	–	+	+	–	–	–	–	+
$f_2(x)$	+	–	+	–	+	–	+	+
$f_3(x)$	–	–	+	–	–	–	–	+
$f_4(x)$	+	+	+	+	+	+	+	+
No. of changes of sign	4	2	0	2	3	2	4	0

Thus the number of real roots = $4 - 0 = 4$ (difference of changes of signs when $x = -\infty$ and $x = \infty$ in the Strum's functions)

Number of negative roots = $2 - 0 = 2$

Number of positive roots = $4 - 2 = 2$

Hence the given equation $f(x) = 0$ has four real roots.

From the above table both positive roots lies in the interval $(2,3)$ and negative roots lies in the intervals $(-2,-1)$ and $(-1,0)$.

Example 8.5.5 :

Find the number and position of real roots of the equation

$$x^4 - 2x^3 + 5x^2 - 4x - 8 = 0$$

Solution : Given that $x^4 - 2x^3 + 5x^2 - 4x - 8 = 0$

Let $f(x) = x^4 - 2x^3 + 5x^2 - 4x - 8$

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Thus $f_1(x) = f'(x) = 4x^3-6x^2+10x-4$

(i.e) $f_1(x) = 2(2x^3-3x^2+5x-2)$

First we shall find, the Strum's functions for $f(x)$.

	$2x^3-3x^2+5x-2$ $\times(-7)$	$x^4-2x^3+5x^2-4x-8$ $\times 2$	
$2x$	$-14x^3+21x^2-35x+14$ $-14x^3+14x^2+68x$	$2x^4-4x^3+10x^2-8x-16$ $2x^4-3x^3+5x^2-2x$	x
-1	$7x^2-103x+14$ $7x^2-7x-34$	$-x^3+5x^2-6x-16$ $\times 2$	
	$-96x+48$ $\div 48$	$-2x^3+10x^2-12x-32$ $-2x^3+3x^2-5x+2$	-1
	$f_3(x)= 2x-1$	$7x^2-7x-34$	
		$f_2(x) = -7x^2+7x+34$ $\times 2$	
		$-14x^2+14x+68$ $-14x^2+7x$	-7
		$7x+68$ $\times 2$	
		$14x+136$ $14x-2$	
		138	
		$f_4(x) = -138$	

Thus the Strum's function of $f(x)$ are given by :

$f(x)=x^4-2x^3+5x^2-4x-8$,

$f_1(x)=2(2x^3-3x^2+5x-2)$,

$f_2(x)=-7x^2+7x+34$,

$f_3(x)=2x-1$,

$f_4(x)=-138$,

The following table shows the signs of Strum's functions by substituting $-$ 0 and ∞ for x.

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I	II	III	IV	V	VI	VII
x	$-\infty$	0	∞	1	-1	2
$f(x)$	+	–	+	–	+	+
$f_1(x)$	–	–	+			
$f_2(x)$	–	+	–			
$f_3(x)$	+	+	–			
$f_4(x)$	–	–	–			
No. of changes of sign	3	2	1			

Thus the number of real roots = $3-1=2$ (difference of changes of signs when $x=-\infty$ and $x=\infty$ in the Strum's functions)

Hence the given equation $f(x)=0$ has two real roots, among one is negative root and the other is positive root, and the remaining two are imaginary roots.

From the above table both positive roots lies in the interval $(1,2)$ and negative roots lies in the intervals $(-1,0)$.

8.6. Cardon's Method

Consider the equation $x^3+px+q=0$ ---------------------------------- (8.40)

Let $x=u+v$

Substituting $x=u+v$ in (8.40), we get,

$(u+v)^3+p(u+v)+q=0$

(i.e) $u^3+v^3+3uv(u+v)+p(u+v)+q=0$

(i.e) $u^3+v^3+q+(u+v)(3uv+p)=0$ ---------------------------------- (8.41)

Now choose u and v in such a way that $3uv+p=0$

Thus (8.41) becomes $u^3+v^3+q=0$ ---------------------------------- (8.42)

With the condition that $3uv+p=0$ ---------------------------------- (8.43)

Now we shall eliminate u from (8.42) and (8.43), we have,

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$$\left(-\frac{p}{3v}\right)^3 + v^3 + q = 0$$

(i.e) $v^6 + qv^3 - \frac{p^3}{27} = 0$ -- (8.44)

Similarly by eliminating v from (8.42) and (8.43), we have,

(i.e) $u^6 + qu^3 - \frac{p^3}{27} = 0$ -- (8.45)

From (8.44) and (8.45), we notice that, u^3 and v^3 are the roots of the equation $t^2 + qt - \frac{p^3}{27} = 0$ -- (8.46)

Now $t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(i.e) $t = \frac{-q \pm \sqrt{q^2 + 4\frac{p^3}{27}}}{2}$

(i.e) $t = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$

(i.e) $u^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$ and $v^3 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$

Roots of (8.46) are real if *discriminent* ≥ 0

(i.e) $\frac{q^2}{4} + \frac{p^3}{27} \geq 0$

In this case the two roots of (8.40) are imaginary and one root is real or all the three roots are real, in which two are equal.

Case (1) : Let the $\frac{q^2}{4} + \frac{p^3}{27} > 0$

Now $\frac{q^2}{4} + \frac{p^3}{27} > 0$

$\Rightarrow 27q^2 + 4p^3 > 0$

Now the roots of (8.46) are real

(i.e) u^3 and v^3 are real.

Let $u^3 = m^3$ and $v^3 = n^3$

Thus u has three values, namely, $m, \omega m, \omega^2 m$ and three values for v, namely, $m, \omega n, \omega^2 n$ where ω is a cube roots of unity.

Hence we have 9 combinations for $u+v$. Out of these 9 combinations, the following 3 combinational values are valid for $u+v$, since $u^3 v^3 = -\frac{p^3}{27}$

$\Rightarrow uv = -\frac{p}{3}$.

(i.e) $m+n$, $m\omega + n\omega^2$, $m\omega^2 + n\omega$ which are the roots of (8.40).

Thus the roots of the cubic equation (8.40) depends on the roots of the equation (8.46)

Case (2) : Let the roots of (8.46) are imaginary

(i.e) $\frac{q^2}{4} + \frac{p^3}{27} < 0$

In this case both u^3 and v^3 are imaginary and hence u and v are cube roots of imaginary quantities. This has no arithmetical meaning.

Hence Cardon's method is not useful.

Example 8.6.1 :

Solve the equation $x^3 - 3x - 65 = 0$ using Cardon's method.

Solution : Given that $x^3 - 3x - 65 = 0$ ---------------------------------- (8.47)

Let $x = u + v$

Now $x = u + v$

$\Rightarrow x^3 = (u+v)^3$

$\Rightarrow x^3 = u^3 + 3u^2 v + 3uv^2 + v^3$

$\Rightarrow x^3 = u^3 + v^3 + 3uv(u+v)$

$\Rightarrow x^3 = u^3 + v^3 + 3uvx$

$\Rightarrow x^3 - 3uvx - (u^3 + v^3) = 0$ -- (8.48)

Comparing (8.47) and (8.48), we have,

$$-3uv = -12 \text{ and } -(u^3 + v^3) = -65$$

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(i.e) $uv = 4$ and $(u^3+v^3) = 65$

Now u^3 and v^3 are the roots of $t^2-(u^3+v^3)t+u^3v^3=0$

(i.e) $t^2-65t+64=0$

(i.e) $t^2-64t-t+64=0$

(i.e) $t(t-64)-(t-64)=0$

(i.e) $(t-64)(t-1)=0$

(i.e) $t=1$ or $t=64$

Let $u^3 = 1$ and $v^3 = 64$

Now $u^3 = 1 \Rightarrow u = 1, \omega, \omega^2$ and

$v^3 = 64 \Rightarrow u = 4, 4\omega, 4\omega^2$ where ω is a cube root of unity.

Thus $x=u+v$, $u\omega+v\omega^2$, $u\omega^2+v\omega$ are the roots of (8.47)

Now $x=u+v \Rightarrow x = 1+4 = 5$,

$$u\omega+v\omega^2 = 1\left(\frac{-1+i\sqrt{3}}{2}\right)+4\left(\frac{-1-i\sqrt{3}}{2}\right)$$

$$= \frac{1}{2}\left(-1+i\sqrt{3}-4-i4\sqrt{3}\right)$$

$$= \frac{1}{2}\left(-5-i3\sqrt{3}\right) \text{ and}$$

$$u\omega^2+v\omega = 1\left(\frac{-1-i\sqrt{3}}{2}\right)+4\left(\frac{-1+i\sqrt{3}}{2}\right)$$

$$= \frac{1}{2}\left(-1-i\sqrt{3}-4+i4\sqrt{3}\right)$$

$$= \frac{1}{2}\left(-5+i3\sqrt{3}\right)$$

Hence the roots of the given equation are $5, \dfrac{-5\pm i3\sqrt{3}}{2}$.

Example 8.6.2 :

Solve the equation $x^3-30x+133=0$ using Cardon's method.

Solution : Given that $x^3-30x+133=0$ ------------------------------- (8.49)

Let $x=u+v$

Now $x = u + v$

$\Rightarrow\ x^3 = (u+v)^3$

$\Rightarrow\ x^3 = u^3 + 3u^2v + 3uv^2 + v^3$

$\Rightarrow\ x^3 = u^3 + v^3 + 3uv(u+v)$

$\Rightarrow\ x^3 = u^3 + v^3 + 3uvx$

$\Rightarrow\ x^3 - 3uvx - (u^3 + v^3) = 0$ -- (8.50)

Comparing (8.49) and (8.50), we have,

$$-3uv = -30 \text{ and } -(u^3 + v^3) = 133$$

(i.e) $uv = 10$ and $(u^3 + v^3) = -133$

Now u^3 and v^3 are the roots of $t^2 - (u^3 + v^3)t + u^3v^3 = 0$

(i.e) $t^2 + 133t + 1000 = 0$

$$\therefore\ t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(i.e) $t = \dfrac{-133 \pm \sqrt{17689 - 4000}}{2}$

(i.e) $t = \dfrac{-133 \pm \sqrt{13689}}{2}$

(i.e) $t = \dfrac{-133 \pm 117}{2}$

(i.e) $t = \dfrac{-133 + 117}{2}$ or $t = \dfrac{-133 - 117}{2}$

(i.e) $t = -8$ or $t = -125$

Let $u^3 = -8$ and $v^3 = -125$

Now $u^3 = -8 \Rightarrow u = -2, -2\omega, -2\omega^2$ and

$v^3 = 64 \Rightarrow u = -5, -5\omega, -5\omega^2$ where ω is a cube root of unity.

Thus $x = u + v$, $u\omega + v\omega^2$, $u\omega^2 + v\omega$ are the roots of (8.49)

Now $x = u + v \Rightarrow x = -2 - 5 = -7$,

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$$u\omega + v\omega^2 = -2\left(\frac{-1+i\sqrt{3}}{2}\right) + (-5)\left(\frac{-1-i\sqrt{3}}{2}\right)$$

$$= \frac{1}{2}\left(2 - i2\sqrt{3} + 5 + i5\sqrt{3}\right)$$

$$= \frac{1}{2}\left(7 + i3\sqrt{3}\right) \text{ and}$$

$$u\omega^2 + v\omega = -2\left(\frac{-1-i\sqrt{3}}{2}\right) + (-5)\left(\frac{-1+i\sqrt{3}}{2}\right)$$

$$= \frac{1}{2}\left(2 + i2\sqrt{3} + 5 - i5\sqrt{3}\right)$$

$$= \frac{1}{2}\left(7 - i3\sqrt{3}\right)$$

Hence the roots of the given equation are $7, \dfrac{7 \pm i3\sqrt{3}}{2}$.

Example 8.6.3 :

Solve the equation $x^3 + 3x - 14 = 0$ using Cardon's method.

Solution : Given that $x^3 + 3x - 14 = 0$ ---------------------------------- (8.51)

Let $x = u + v$

Now $x = u + v$

$\Rightarrow x^3 = (u+v)^3$

$\Rightarrow x^3 = u^3 + 3u^2v + 3uv^2 + v^3$

$\Rightarrow x^3 = u^3 + v^3 + 3uv(u+v)$

$\Rightarrow x^3 = u^3 + v^3 + 3uvx$

$\Rightarrow x^3 - 3uvx - (u^3 + v^3) = 0$ -- (8.52)

Comparing (8.51) and (8.52), we have,

$$-3uv = 3 \text{ and } -(u^3 + v^3) = -14$$

(i.e) $uv = -1$ and $(u^3 + v^3) = 14$

Now u^3 and v^3 are the roots of $t^2 - (u^3 + v^3)t + u^3v^3 = 0$

(i.e) $t^2 - 14t - 1 = 0$

$$\therefore\ t=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\text{(i.e) } t=\frac{14\pm\sqrt{196+4}}{2}$$

$$\text{(i.e) } t=\frac{14\pm\sqrt{200}}{2}$$

$$\text{(i.e) } t=\frac{14\pm 10\sqrt{2}}{2}$$

(i.e) $t=7+5\sqrt{2}$ or $t=7-5\sqrt{2}$

Let $u^3=-8$ and $v^3=-125$

Now $u^3=7+5\sqrt{2}$ and $v^3=7-5\sqrt{2}$

Let $u=a+b\sqrt{2}$

Now $u=a+b\sqrt{2}$

$$\Rightarrow u^3=\left(a+b\sqrt{2}\right)^3$$

$$\Rightarrow \left(a+b\sqrt{2}\right)^3=u^3$$

$$\Rightarrow a^3+3\sqrt{2}\,a^2b+6ab^2+2\sqrt{2}\,b^3=7+5\sqrt{2}$$

Comparing rational and irrational coefficients on both sides, we get,

$a^3+6a^2=7$ and $3a^2b+2b^3=5$

$a(a^2+6a)=7$ -- (8.53)

and $b(3a^2+2b^2)=5$ -- (8.54)

From (8.53) and (8.54) it is clear that a, b are factors of 7 and 5 respectively.

Using inspection method, we have, $a=b=1$

Thus $u=1+\sqrt{2}$ and $v=1-\sqrt{2}$

Now $x=u+v \Rightarrow x=1+\sqrt{2}+1-\sqrt{2}=2$

To find the other roots,

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2	1	0	3	-14
	0	2	4	14
	1	2	7	0

Now $x^3+3x-14=(x-2)(x^2+2x+7)$

Again $x^3+3x-14=0$

$\Rightarrow (x-2)(x^2+2x+7)=0$

$\Rightarrow x-2=0$ or $x^2+2x+7=0$

$\Rightarrow x=2$ or $x^2+2x+7=0$

Now $x^2+2x+7=0$

$$\Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow x=\frac{-2\pm\sqrt{4-28}}{2}$$

$$\Rightarrow x=\frac{-2\pm\sqrt{-24}}{2}$$

$$\Rightarrow x=\frac{-2\pm i2\sqrt{6}}{2}$$

$$\Rightarrow x=-1\pm i\sqrt{6}$$

Hence the roots of the given equation (8.51) are $2, -1\pm i\sqrt{6}$.

Example 8.6.4 :

Solve the equation $28x^3-9x+1=0$ using Cardon's method.

Solution : Given that $28x^3-9x+1=0$ ------------------------------------ (8.55)

Put $x=\frac{1}{y}$ in (8.55), we have, $28\left(\frac{1}{y}\right)^3-9\left(\frac{1}{y}\right)+1=0$

(i.e) $y^3-9y+28=0$ -- (8.56)

Let $y=u+v$

Now $y = u + v$

$\Rightarrow y^3 = (u+v)^3$

$\Rightarrow y^3 = u^3 + 3u^2v + 3uv^2 + v^3$

$\Rightarrow y^3 = u^3 + v^3 + 3uv(u+v)$

$\Rightarrow y^3 = u^3 + v^3 + 3uvy$

$\Rightarrow y^3 - 3uvy - (u^3 + v^3) = 0$ --- (8.57)

Comparing (8.56) and (8.57), we have,

$-3uv = -9$ and $-(u^3 + v^3) = 28$

(i.e) $uv = 3$ and $(u^3 + v^3) = -28$

Now u^3 and v^3 are the roots of $t^2 - (u^3 + v^3)t + u^3v^3 = 0$

(i.e) $t^2 + 28t + 27 = 0$

(i.e) $t^2 + 27t + t + 27 = 0$

(i.e) $t(t+27) + (t+27) = 0$

(i.e) $(t+27)(t+1) = 0$

(i.e) $t = -1$ or $t = -27$

Let $u^3 = -1$ and $v^3 = -27$

Thus one real value of $u = -1$

and one real value of $v = -1$

Now $y = u + v = -1 - 3 = -3$

(i.e) $x = \dfrac{1}{y} = -\,4$

(i.e) one root of the given equation (8.55) is $-\dfrac{1}{4}$.

To find the other roots, we may use the synthetic division

- 1/4	28	-9	0	1
	0	-7	4	-1
	28	-16	4	0

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Now $28x^3 - 9x + 1 = \left(x + \frac{1}{4}\right)\left(28x^2 - 16x + 14\right)$

Again $28x^3 - 9x + 1 = 0$

$\Rightarrow \left(x + \frac{1}{4}\right)\left(28x^2 - 16x + 14\right)$

$\Rightarrow x + \frac{1}{4} = 0$ or $28x^2 - 16x + 14 = 0$

Now $28x^2 - 16x + 14 = 0$

$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow x = \frac{4 \pm \sqrt{16 - 28}}{14}$

$\Rightarrow x = \frac{4 \pm \sqrt{-12}}{14}$

$\Rightarrow x = \frac{4 \pm i2\sqrt{3}}{2}$

$\Rightarrow x = \frac{1}{7}\left(2 \pm i\sqrt{3}\right)$

Hence the roots of the given equation are $-\frac{1}{4}, \frac{1}{7}\left(2 \pm i\sqrt{3}\right)$.

Example 8.6.5 :

Solve the equation $x^3 + 6x^2 + 9x + 4 = 0$ using Cardon's method.

Solution : Given that $x^3 + 6x^2 + 9x + 4 = 0$ --------------------------- (8.58)

First we shall remove the second term in the given equation (8.58)

Put $x = y + h$ in (8.58), we have,

$(y+h)^3 + 6(y+h)^2 + 9(y+h) + 4 = 0$

(i.e) $y^3 + (3h+6)y^2 + \text{L} = 0$

If $3h + 6 = 0$ then $h = -2$

$\therefore$ Diminish the roots of (8.58) by 2 and the synthetic division is show below

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-2	1	6	9	4
	0	-2	-8	-2
	1	4	1	2
	0	-2	-4	
	1	2	-3	
	0	-2		
	1	0		

Thus the transformed equation is $y^3 - 3y + 2 = 0$ ---------------------- (8.59)

Let $y = u + v$

Now $y = u + v$

$\Rightarrow \ y^3 = (u+v)^3$

$\Rightarrow \ y^3 = u^3 + 3u^2v + 3uv^2 + v^3$

$\Rightarrow \ y^3 = u^3 + v^3 + 3uv(u+v)$

$\Rightarrow \ y^3 = u^3 + v^3 + 3uvy$

$\Rightarrow \ y^3 - 3uvy - (u^3 + v^3) = 0$ -- (8.60)

Comparing (8.59) and (8.60), we have,

$$-3uv = -9 \text{ and } -(u^3 + v^3) = 2$$

(i.e) $uv = 1$ and $(u^3 + v^3) = -2$

Now u^3 and v^3 are the roots of $t^2 - (u^3 + v^3)t + u^3v^3 = 0$

(i.e) $t^2 + 2t + 1 = 0$

(i.e) $(t+1)^2 = 0$

(i.e) $t = -1, -1$

Let $u^3 = -1$ and $v^3 = -1$

Now $u^3 = -1 \Rightarrow u = -1, -\omega, -\omega^2$ and

$v^3 = -1 \Rightarrow v = -1, -\omega, -\omega^2$ where ω is a cube root of unity.

Thus $x = u + v,\ u\omega + v\omega^2,\ u\omega^2 + v\omega$ are the roots of (8.59)

Now $x = u + v \Rightarrow x = -1 - 1 = -2,$

$u\omega + v\omega^2 = -\omega - \omega^2 = 1$ (Q $1 + \omega + \omega^2 = 0$)

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$u\omega^2 + v\omega = -\omega^2 - \omega = 1$

(i.e) the roots of the equation (8.59) are $-2, 1, 1$

Hence the roots of the given equation (8.58) are $-2-2, 1-2, 1-2$.

(i.e) $-4, -1, -1$ are the required roots of the given equation

Example 8.6.6 :

Solve the equation $27x^3 + 54x^2 + 198x - 73 = 0$ using Cardon's method.

Solution : Given that $27x^3 + 54x^2 + 198x - 73 = 0$ ------------------ (8.61)

First we shall remove the second term in the given equation (8.61)

Put $x = y + h$ in (8.61), we have,

$$27(y+h)^3 + 54(y+h)^2 + 198(y+h) - 73 = 0$$

(i.e) $27y^3 + (81h + 54)y^2 + \text{L} = 0$

If $81h + 54 = 0$ then $h = -\frac{54}{81} = -\frac{2}{3}$

$\therefore$ Diminish the roots of (8.61) by $-\frac{2}{3}$ and the synthetic division is show below:

- 2/3	27	54	198	-73
	0	-18	-24	-116
	27	36	174	-189
	0	-18	-12	
	27	18	162	
	0	-18		
	27	0		

Thus the transformed equation is $27z^3 + 162z - 189 = 0$

(i.e) $z^3 + 6z - 7 = 0$ -- (8.62)

Let $z = u + v$

Now $z = u + v$

$\Rightarrow y^3 = (u+v)^3$

$\Rightarrow z^3 = u^3 + 3u^2v + 3uv^2 + v^3$

$\Rightarrow z^3 = u^3 + v^3 + 3uv(u+v)$

$$\Rightarrow z^3 = u^3 + v^3 + 3uvz$$

$$\Rightarrow z^3 - 3uvz - (u^3 + v^3) = 0 \quad \text{-----------------------------------} \quad (8.63)$$

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Comparing (8.62) and (8.63), we have,

$$-3uv = 6 \text{ and } -(u^3 + v^3) = -7$$

(i.e) $uv = -2$ and $(u^3 + v^3) = 7$

Now u^3 and v^3 are the roots of $t^2 - (u^3 + v^3)t + u^3v^3 = 0$

(i.e) $t^2 - 7t - 8 = 0$

$$\therefore t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow t = \frac{7 \pm \sqrt{49 + 32}}{2}$$

$$\Rightarrow t = \frac{7 \pm 9}{2}$$

$$\Rightarrow t = 8 \text{ or } t = -1$$

Let $u^3 = -1$ and $v^3 = 8$

Now $u^3 = -1 \Rightarrow u = -1, -\omega, -\omega^2$ and

$v^3 = 8 \Rightarrow v = 2$ where ω is a cube root of unity.

Now $x = u + v \Rightarrow x = -1 + 2 = 1$

$$u\omega + v\omega^2 = -\omega + 2\omega^2$$

$$= -\frac{-1 + i\sqrt{3}}{2} + 2\frac{-1 - i\sqrt{3}}{2}$$

$$= \frac{+1 - i\sqrt{3}}{2} + \frac{-2 - i\,2\sqrt{3}}{2}$$

$$= \frac{1}{2}\left[\frac{-1 - i3\sqrt{3}}{2}\right]$$

$$u\omega^2 + v\omega = -\omega^2 + 2\omega$$

$$= -\frac{-1 - i\sqrt{3}}{2} + 2\frac{-1 + i\sqrt{3}}{2}$$

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$$= \frac{+1+i\sqrt{3}}{2} + \frac{-2+i\,2\sqrt{3}}{2}$$

$$= \frac{1}{2}\left[\frac{-1+i3\sqrt{3}}{2}\right]$$

(i.e) the roots of the equation (8.63) are 1, $\frac{1}{2}\left[\frac{-1+i3\sqrt{3}}{2}\right]$

Hence the roots of the given equation (8.61) are $\frac{1}{3}, \frac{1}{6}\left[\frac{-7+i9\sqrt{3}}{2}\right]$.

Check your progress

Question

Using Cardon's method, solve the following equations.

(1) $x^3 - 15x^2 - 157x + 5491 = 0$ (Answer : -19, 17, 17)

(2) $x^3 - 3x + 1 = 0$ (Answer : $2\cos\frac{2\pi}{9}, 2\cos\frac{8\pi}{9}, 2\cos\frac{14\pi}{9}$)

Summary

In this unit we have learned the method of finding roots of an equation by removing a particular term, using Descarte's rule of sign and Roll's theorem we learned the nature roots of an equation. Then we discussed multiple roots of an equation and using Strum's theorem we learned that the number and position of roots of an equation. Finally, we discussed the method solving a cubic equation using Cardon's method.

Further Reading

You can also refer the following books for further reading.

(1) Algebra by T.K.Manicavachagom Pillai and others Vol I

(2) Classical Algebra by Arumugam and others.

UNIT IX

FERRARI'S METHOD

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Unit Objectives

Unit Structure

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Summary

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Objectives :

In this unit, we are going to discuss to find roots of a biquadratic equation using Ferrari's method, expansion of $\sin n\theta$, $\cos n\theta$, $\tan n\theta$, $\sin^n\theta$, $\cos^n\theta$, $\tan^n\theta$, $\sin^m\theta\cos^n\theta$.

After completing this unit, students may able to know

- o to find roots of a biquadratic equation using Ferrari's method
- o expansion of $\sin n\theta$, $\cos n\theta$, $\tan n\theta$, $\sin^n\theta$, $\cos^n\theta$, $\tan^n\theta$, $\sin^m\theta\cos^n\theta$

9.1. Ferrari's Method

In this section, we are going to discuss to find roots of a biquadratic equation using Ferrari's method. This method consists of expressing the given biquadratic equation as a difference of two perfect squares and factorizing it into tow quadratic expressions.

Let the biquadratic equation be $x^4+2ax^3+bx^2+cx+d=0$ ----- (9.1)

Now $x^4+2ax^3+bx^2+cx+d=0$

$\Rightarrow$ $x^4+2ax^3 = -(bx^2+cx+d)$

$\Rightarrow (x^2+ax+\lambda)^2 = -bx^2-cx-d+2\lambda x^2+a^2x^2+\lambda^2+2a\lambda x$

$\Rightarrow (x^2+ax+\lambda)^2 = (a^2+2\lambda-b)x^2+(2a\lambda-c)x+(\lambda^2-d)$ ----- (9.2)

Select that value of λ so that the RHS of (9.2) becomes a perfect square.

Now the RHS of (9.2) is perfect square if its discriminent is zero.

(i.e) $(2a\lambda-c)^2-4(a^2+2\lambda-b)(\lambda^2-d)=0$ ------------------------- (9.3)

Let λ_1, λ_2, λ_3 be the roots of (9.3)

Substituting $\lambda=\lambda_1$ in (9.2), after simplification, let the equation becomes

$(x^2+ax+\lambda)^2=(px+q)^2$

which is the difference of two square factors.

Hence it can be factored into two quadratic factors, which in turn give rise to complete factorization. The other two values of λ also gives the same factorization and hence give the same set of solutions to the given equation. Thus it is enough to find any root of the cubic equation in λ.

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Example 9.1.1 :

Solve the equation $x^4+4x^3-x^2+8x+4=0$ using Ferrari's method

Solution :

Given that $x^4+4x^3-x^2+8x+4=0$ ----------------------------------- (9.4)

(i.e) $(x^2+2x+\lambda)^2-4x^2-\lambda^2-2\lambda x^2-4\lambda x-x^2+8x+4=0$

(i.e) $(x^2+2x+\lambda)^2-[4x^2+\lambda^2+2\lambda x^2+4\lambda x+x^2-8x-4]=0$

(i.e) $(x^2+2x+\lambda)^2=4x^2+\lambda^2+2\lambda x^2+4\lambda x+x^2-8x-4$

(i.e) $(x^2+2x+\lambda)^2=(5+2\lambda)x^2+(4\lambda-8)x+(\lambda^2-4)$ ------------- (9.5)

Now RHS of (9.5) becomes perfect square if its discriminent is zero

(i.e) $(4\lambda-8)^2-4(5+2\lambda)(\lambda^2-4)=0$

(i.e) $4(\lambda-2)^2-4(5+2\lambda)(\lambda+2)(\lambda-2)=0$

Clearly $\lambda=2$ is a root of the above equation.

Put $\lambda=2$ in (9.5), we have, $(x^2+2x+2)^2=(5+4)x^2+(8-8)x+(4-4)$

(i.e) $(x^2+2x+2)^2=9x^2$

(i.e) $(x^2+2x+2)^2-9x^2=0$

(i.e) $(x^2+2x+2)^2-(3x)^2=0$

(i.e) $(x^2+2x+2+3x)(x^2+2x+2-3x)=0$

(i.e) $(x^2+5x+2)(x^2-x+2)=0$

(i.e) $x^2+5x+2=0$ or $x^2-x+2=0$

If $x^2+5x+2=0$ then $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

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$$\Rightarrow x = \frac{-5 \pm \sqrt{25-8}}{2}$$

$$\Rightarrow x = \frac{-5 \pm \sqrt{17}}{2}$$

If $x^2 - x + 2 = 0$ then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\Rightarrow x = \frac{1 \pm \sqrt{1-8}}{2}$$

$$\Rightarrow x = \frac{1 \pm i\sqrt{7}}{2}$$

Thus the roots of the given equation are $\frac{-5 \pm \sqrt{17}}{2}, \frac{1 \pm i\sqrt{7}}{2}$

Example 9.1.2 :

Solve the equation $x^4 - 3x^2 - 42x - 40 = 0$ using Ferrari's method

Solution :

Given that $x^4 - 3x^2 - 42x - 40 = 0$ -- (9.6)

(i.e) $(x^2 + \lambda)^2 - \lambda^2 - 2\lambda x^2 - 3x^2 - 42x - 40 = 0$

(i.e) $(x^2 + \lambda)^2 - [\lambda^2 + 2\lambda x^2 + 3x^2 + 42x + 40] = 0$

(i.e) $(x^2 + \lambda)^2 = [\lambda^2 + 2\lambda x^2 + 3x^2 + 42x + 40]$

(i.e) $(x^2 + \lambda)^2 = (2\lambda + 3)x^2 + 42x + (\lambda^2 + 40)$ ------------------------ (9.7)

Now RHS of (9.7) becomes perfect square if its discriminent is zero

(i.e) $(42)^2 - 4(2\lambda + 3)(\lambda^2 + 40) = 0$

(i.e) $441 - (2\lambda^3 + 80\lambda + 3\lambda^2 + 120) = 0$

(i.e) $2\lambda^3 + 3\lambda^2 + 80\lambda - 321 = 0$

By inspection, $\lambda = 3$ is a root of the above equation.

Put $\lambda = 3$ in (9.7), we have, $(x^2 + 3)^2 = (6+3)x^2 + 42x + (9+40)$

(i.e) $(x^2 + 3)^2 = 9x^2 + 42x + 49$

(i.e) $(x^2 + 3)^2 = (3x + 7)^2$

(i.e) $(x^2+3)^2-(3x+7)^2=0$

(i.e) $(x^2+3+3x+7)(x^2+3-(3x+7))=0$

(i.e) $(x^2+3x+10)(x^2-3x-4)=0$

(i.e) $x^2+3x+10=0$ or $x^2-3x-4=0$

If $x^2+3x+10=0$ then $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\Rightarrow\ x=\dfrac{-3\pm\sqrt{9-40}}{2}$

$\Rightarrow\ x=\dfrac{-3\pm i\sqrt{31}}{2}$

If $x^2-3x-4=0$ then $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\Rightarrow\ x=\dfrac{3\pm\sqrt{9+16}}{2}$

$\Rightarrow\ x=\dfrac{3\pm 5}{2}$

$\Rightarrow\ x=\dfrac{3+5}{2}$ or $x=\dfrac{3-5}{2}$

$\Rightarrow\ x=4$ or $x=-1$

Thus the roots of the given equation are $4, -1, \dfrac{-3\pm i\sqrt{31}}{2}$.

Example 9.1.3 :

Solve the equation $x^4-10x^3+35x^2-50x+24=0$ using Ferrari's method

Solution :

Given that $x^4-10x^3+35x^2-50x+24=0$ ---------------------------- (9.8)

(i.e) $(x^2-5x+\lambda)^2-25x^2-\lambda^2-2\lambda x^2+10\lambda x+35x^2-50x+24=0$

(i.e) $(x^2-5x+\lambda)^2-[25x^2+\lambda^2+2\lambda x^2-10\lambda x-35x^2+50x-24]=0$

(i.e) $(x^2-5x+\lambda)^2=25x^2+\lambda^2+2\lambda x^2-10\lambda x-35x^2+50x-24$

(i.e) $(x^2-5x+\lambda)^2=(2\lambda-10)x^2+10(5-\lambda)x+(\lambda^2-24)$ --------- (9.9)

Now RHS of (9.9) becomes perfect square if its discriminent is zero

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Space for Hints

(i.e) $100(5-\lambda)^2-4(2\lambda-10)(\lambda^2-24)=0$

Clearly $\lambda=5$ is a root of the above equation.

Put $\lambda=5$ in (9.9), we have,

$(x^2-5x+5)^2=(10-10)x^2+10(5-5)x+(25-24)$

(i.e) $(x^2-5x+5)^2=1$

(i.e) $x^2-5x+5=\pm1$

(i.e) $x^2-5x+6=0$ or $x^2-5x+4=0$

If $x^2-5x+6=0$ then $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\Rightarrow\ x=\dfrac{5\pm\sqrt{25-24}}{2}$

$\Rightarrow\ x=\dfrac{5\pm1}{2}$

$\Rightarrow\ x=3$ or $x=2$

If $x^2-5x+4=0$ then $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\Rightarrow\ x=\dfrac{5\pm\sqrt{25-16}}{2}$

$\Rightarrow\ x=\dfrac{5\pm3}{2}$

$\Rightarrow\ x=4$ or $x=1$

Thus the roots of the given equation are 1, 2, 3, 4.

Example 9.1.4 :

Solve the equation $x^4-8x^3-12x^2+60x+63=0$ using Ferrari's method

Solution :

Given that $x^4-8x^3-12x^2+60x+63=0$ ---------------------------- (9.10)

(i.e) $(x^2-4x+\lambda)^2-16x^2-\lambda^2-2\lambda x^2+8\lambda x-12x^2+60x+63=0$

(i.e) $(x^2-4x+\lambda)^2-[16x^2+\lambda^2+2\lambda x^2-8\lambda x+12x^2-60x-63]=0$

(i.e) $(x^2-4x+\lambda)^2=16x^2+\lambda^2+2\lambda x^2-8\lambda x+12x^2-60x-63$

(i.e) $(x^2-4x+\lambda)^2=(2\lambda+28)x^2-4(2\lambda+15)x+(\lambda^2-63)$ ------ (9.11)

Now RHS of (9.11) becomes perfect square if its discriminent is zero

(i.e) $16(2\lambda+15)^2-4(2\lambda+28)(\lambda^2-63)=0$

(i.e) $2(2\lambda+15)^2-(\lambda+14)(\lambda^2-63)=0$

(i.e) $-\lambda^3-6\lambda^2+183\lambda+1332=0$

By inspection $\lambda=-12$ is a root of the above equation.

Put $\lambda=-12$ in (9.11), we have,

$(x^2-4x-12)^2=(-24+28)x^2-4(-24+15)x+(144-63)$

(i.e) $(x^2-4x-12)^2=4x^2+36x+81$

(i.e) $(x^2-4x-12)^2=(2x+9)^2$

(i.e) $(x^2-4x-12)^2-(2x+9)^2=0$

$(x^2-4x-12+2x+9)(x^2-4x-12-(2x+9))=0$

$(x^2-2x-3)(x^2-6x-21)=0$

(i.e) $x^2-2x-3=0$ or $x^2-6x-21=0$

If $x^2-2x-3=0$ then $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\Rightarrow x=\dfrac{2\pm\sqrt{4+12}}{2}$

$\Rightarrow x=\dfrac{2\pm 4}{2}$

$\Rightarrow x=3$ or $x=-1$

If $x^2-6x-21=0$ then $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\Rightarrow x=\dfrac{6\pm\sqrt{36+84}}{2}$

$\Rightarrow x=\dfrac{6\pm\sqrt{120}}{2}$

$\Rightarrow x=\dfrac{6\pm 2\sqrt{30}}{2}$

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Space for Hints

$\Rightarrow \ x = 3 \pm \sqrt{30}$

Thus the roots of the given equation are $-1, 3, 3 \pm \sqrt{30}$.

Example 9.1.5 :

Solve the equation $x^4 - 18x^2 + 32x - 15 = 0$ using Ferrari's method

Solution :

Given that $x^4 - 18x^2 + 32x - 15 = 0$ ------------------------------------ (9.12)

(i.e) $(x^2 + \lambda)^2 - \lambda^2 - 2\lambda x^2 - 18x^2 + 32x - 15 = 0$

(i.e) $(x^2 + \lambda)^2 - [\lambda^2 + 2\lambda x^2 + 18x^2 - 32x + 15] = 0$

(i.e) $(x^2 + \lambda)^2 = [\lambda^2 + 2\lambda x^2 + 18x^2 - 32x + 15]$

(i.e) $(x^2 + \lambda)^2 = 2(\lambda + 9)x^2 - 32x + (\lambda^2 + 15)$ ------------------------ (9.13)

Now RHS of (9.13) becomes perfect square if its discriminent is zero

(i.e) $(32)^2 - 4 \times 2 \times (\lambda + 9)(\lambda^2 + 15) = 0$

(i.e) $128 - (\lambda^3 + 15\lambda + 9\lambda^2 + 135) = 0$

(i.e) $\lambda^3 + 9\lambda^2 + 15\lambda + 7 = 0$

By inspection, $\lambda = -1$ is a root of the above equation.

Put $\lambda = -1$ in (9.13), we have, $(x^2 - 1)^2 = 2(-1+9)x^2 - 32x + (1+15)$

(i.e) $(x^2 - 1)^2 = 2(8)x^2 - 32x + (16)$

(i.e) $(x^2 - 1)^2 = (4x - 4)^2$

(i.e) $(x^2 - 1)^2 - (4x - 4)^2 = 0$

(i.e) $(x^2 - 1 + 4x - 4)(x^2 + 3 - (4x - 4)) = 0$

(i.e) $(x^2 + 4x - 5)(x^2 - 4x + 3) = 0$

(i.e) $x^2 + 4x - 5 = 0$ or $x^2 - 4x + 3 = 0$

If $x^2 + 4x - 5 = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow \ x = \dfrac{-4 \pm \sqrt{16 + 20}}{2}$

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$$\Rightarrow\ x=\frac{-4\pm 6}{2}$$

$$\Rightarrow\ x=\frac{-4-6}{2}\ \text{or}\ x=\frac{-4+6}{2}$$

$$\Rightarrow\ x=-5\ \text{or}\ x=1$$

If $x^2-4x+3=0$ then $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$\Rightarrow\ x=\frac{4\pm\sqrt{16-12}}{2}$$

$$\Rightarrow\ x=\frac{4\pm 2}{2}$$

$$\Rightarrow\ x=\frac{4+2}{2}\ \text{or}\ x=\frac{4-2}{2}$$

$$\Rightarrow\ x=3\ \text{or}\ x=1$$

Thus the roots of the given equation are $1, 1, 3, -5$.

Check your progress

Questions :

Solve the following equations using Ferrari's method.

(1) $x^4+8x^3+12x^2+4x+5=0$ (Answer : $\dfrac{-2\pm i\sqrt{6}}{2}, \pm\dfrac{i}{\sqrt{2}}$)

(2) $x^4-2x^2+8x-3=0$ (Answer : $-1\pm\sqrt{2}, 1\pm i\sqrt{2}$)

(3) $x^4+2x^3-7x^2-8x+12=0$ (Answer : $1, 2, -2, -3$)

9.2. Expansion of $\sin n\theta$, $\cos n\theta$, $\tan n\theta$

Example 9.2.1 :

Find the expansion of $\cos n\theta$ and $\sin n\theta$ in powers of $\sin\theta$ and $\cos\theta$

Solution : We know that $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$

Space for Hints

$$= \cos^n\theta + {}^nC_1\cos^{n-1}\theta(i\sin\theta) + {}^nC_2\cos^{n-2}\theta(i\sin\theta)^2$$

$$+ {}^nC_3\cos^{n-3}\theta(i\sin\theta)^3 + {}^nC_4\cos^{n-4}\theta(i\sin\theta)^4 + \mathrm{L}$$

$$= \cos^n\theta + i\,{}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta$$

$$+-i\cdot{}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta + \mathrm{L}$$

Equating real and imaginary parts on both sides, we have,

$$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta + \mathrm{L}$$

and

$$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta\mathrm{L}$$

which are the required expansions of $\cos n\theta$ and $\sin n\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$.

Example 9.2.2 :

Find the expansion of $\tan n\theta$.

Solution :

We know that

$$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta + \mathrm{L}$$

and

$$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta\mathrm{L}$$

Now $\tan n\theta = \dfrac{\sin n\theta}{\cos n\theta}$

$$= \frac{{}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta\mathrm{L}}{\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta + \mathrm{L}}$$

$$= \frac{\cos^n\theta\left({}^nC_1\tan\theta - {}^nC_2\tan^2\theta + {}^nC_5\tan^5\theta - \mathrm{L}\right)}{\cos^n\theta\left(1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - {}^nC_6\tan^6\theta - \mathrm{L}\right)}$$

$$= \frac{{}^nC_1\tan\theta - {}^nC_2\tan^2\theta + {}^nC_5\tan^5\theta - \mathrm{L}}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - {}^nC_6\tan^6\theta - \mathrm{L}}$$

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Thus $\tan n\theta = \dfrac{{}^nC_1 \tan\theta - {}^nC_2 \tan^2\theta + {}^nC_5 \tan^5\theta - \text{L}}{1 - {}^nC_2 \tan^2\theta + {}^nC_4 \tan^4\theta - {}^nC_6 \tan^6\theta - \text{L}}$

Example 9.2.3 :

Expand $\cos 5\theta$

Solution :

We know that $\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$

$$= \cos^5\theta + {}^5C_1 \cos^{5-1}\theta(i\sin\theta) + {}^5C_2 \cos^{5-2}\theta(i\sin\theta)^2$$

$$+ {}^5C_3 \cos^{5-3}\theta(i\sin\theta)^3 + {}^5C_4 \cos^{5-4}\theta(i\sin\theta)^4$$

$$+ {}^5C_5 \cos^{5-5}\theta(i\sin\theta)^5$$

$$= \cos^5\theta + i5\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - i10\cos^2\theta\sin^3\theta$$

$$+ 5\cos\theta\sin^4\theta + i\sin^5\theta$$

Equating real parts on both sides, we have,

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

Example 9.2.4 :

Expand $\sin 7\theta$ in powers of $\cos\theta$ and $\sin\theta$. Also, deduce that

$$\frac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$$

Solution :

We know that $\cos 7\theta + i\sin 7\theta = (\cos\theta + i\sin\theta)^7$

$$= \cos^7\theta + {}^7C_1 \cos^6\theta(i\sin\theta) + {}^7C_2 \cos^5\theta(i\sin\theta)^2$$

$$+ {}^7C_3 \cos^4\theta(i\sin\theta)^3 + {}^7C_4 \cos^3\theta(i\sin\theta)^4$$

$$+ {}^7C_5 \cos^2\theta(i\sin\theta)^5 + {}^7C_6 \cos\theta(i\sin\theta)^6 + {}^7C_7 (i\sin\theta)^7$$

$$= \cos^7\theta + i7\cos^6\theta\sin\theta - 21\cos^5\theta\sin^2\theta - i35\cos^4\theta\sin^3\theta$$

$$+ 35\cos^3\theta\sin^4\theta + 21i\cos^2\theta\sin^5\theta - 7\cos\theta\sin^6\theta - i\sin^7\theta$$

Space for Hints

Equating imaginary parts on both sides, we have,

$$\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$$

Deduction :

$$\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$$

$$\text{(i.e)}\ \frac{\sin 7\theta}{\sin\theta} = 7\cos^6\theta - 35\cos^4\theta\sin^2\theta + 21\cos^2\theta\sin^4\theta - \sin^6\theta$$

$$= 7(1-\sin^2\theta)^3 - 35(1-\sin^2\theta)^2\sin^2\theta + 21(1-\sin^2\theta)\sin^4\theta - \sin^6\theta$$

$$= 7(1-3\sin^2\theta+3\sin^4\theta-\sin^6\theta) - 35\sin^2\theta(1-2\sin^2\theta+\sin^4\theta)$$

$$+ 21\sin^4\theta(1-\sin^2\theta) - \sin^6\theta$$

$$= 7 + (-21-35)\sin^2\theta + (21+70+21)\sin^4\theta + (-7-35-21-1)\sin^6\theta$$

$$= 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$$

$$\text{Thus}\ \frac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$$

This proves the problem.

Check your progress

Question :

Prove that $\dfrac{\sin 9\theta}{\sin\theta} = 256\cos^8\theta - 448\cos^6\theta + 240\cos^4\theta - 49\cos^2\theta + 1$.

Example 9.2.5 :

If α, β, γ are the roots of the equation $x^3 + px^2 + qx + p = 0$, prove that

$\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = n\pi$ radians except when $q = 1$.

Solution :

Given that α, β, γ are the roots of the equation

$x^3 + px^2 + qx + p = 0$

$\therefore\ \sum\alpha = -p$ -- (9.14)

$\sum\alpha\beta = q$ -- (9.15)

$\alpha\beta\gamma = -p$ -- (9.16)

Let $x_1 = \tan^{-1}\alpha$, $x_2 = \tan^{-1}\beta$, $x_3 = \tan^{-1}\gamma$

$\therefore\ \alpha = \tan x_1$, $\beta = \tan x_2$, $\gamma = \tan x_3$

From (9.14), (9.15) and (9.16), we have,

$S_1 = -p$, $S_2 = q$ and $S_3 = -p$

We know that $\tan(x_1 + x_2 + x_3) = \dfrac{S_1 - S_3 + S_5 - \text{L}}{1 - S_2 + S_4 - \text{L}}$

(i.e) $\tan(x_1 + x_2 + x_3) = \dfrac{-p + p}{1 - q}$

(i.e) $x_1 + x_2 + x_3 = \tan^{-1}(0)$ when $q \neq 1$

(i.e) $\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = n\pi$

This proves the problem.

Example 9.2.6 :

Expand $\sin 7\theta$ as a polynomial in $\sin\theta$. Hence obtain the cubic equation whose roots are $\sin^2\left(\dfrac{2\pi}{7}\right)$, $\sin^2\left(\dfrac{4\pi}{7}\right)$, $\sin^2\left(\dfrac{6\pi}{7}\right)$.

Solution :

Step 1 : First we shall find the expansion of $\sin 7\theta$.

We know that $\cos 7\theta + i\sin 7\theta = (\cos\theta + i\sin\theta)^7$

$$= \cos^7\theta + {}^7C_1\cos^6\theta(i\sin\theta) + {}^7C_2\cos^5\theta(i\sin\theta)^2$$

$$+ {}^7C_3\cos^4\theta(i\sin\theta)^3 + {}^7C_4\cos^3\theta(i\sin\theta)^4$$

$$+ {}^7C_5\cos^2\theta(i\sin\theta)^5 + {}^7C_6\cos\theta(i\sin\theta)^6 + {}^7C_7(i\sin\theta)^7$$

$$= \cos^7\theta + i7\cos^6\theta\sin\theta - 21\cos^5\theta\sin^2\theta - i35\cos^4\theta\sin^3\theta$$

$$+ 35\cos^3\theta\sin^4\theta + 21i\cos^2\theta\sin^5\theta - 7\cos\theta\sin^6\theta - i\sin^7\theta$$

Equating imaginary parts on both sides, we have,

$$\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$$

Now $\dfrac{\sin 7\theta}{\sin\theta} = 7\cos^6\theta - 35\cos^4\theta\sin^2\theta + 21\cos^2\theta\sin^4\theta - \sin^6\theta$

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$$= 7(1-\sin^2\theta)^3 - 35(1-\sin^2\theta)^2\sin^2\theta + 21(1-\sin^2\theta)\sin^4\theta - \sin^6\theta$$

$$= 7(1-3\sin^2\theta+3\sin^4\theta-\sin^6\theta) - 35\sin^2\theta(1-2\sin^2\theta+\sin^4\theta)$$

$$+ 21\sin^4\theta(1-\sin^2\theta) - \sin^6\theta$$

$$= 7 + (-21-35)\sin^2\theta + (21+70+21)\sin^4\theta + (-7-35-21-1)\sin^6\theta$$

$$= 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$$

Thus $\dfrac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$

Hence $\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$ ----------- (9.17)

Step 2 : Now we shall find the cubic equation whose roots are

$\sin^2\left(\dfrac{2\pi}{7}\right)$, $\sin^2\left(\dfrac{4\pi}{7}\right)$, $\sin^2\left(\dfrac{6\pi}{7}\right)$.

If θ is one of the following values $0, \dfrac{2\pi}{7}, \dfrac{4\pi}{7}, \dfrac{6\pi}{7}, \dfrac{8\pi}{7}, \dfrac{10\pi}{7}, \dfrac{12\pi}{7}$, then

$\sin 7\theta = 0$

Hence $0, \dfrac{2\pi}{7}, \dfrac{4\pi}{7}, \dfrac{6\pi}{7}, \dfrac{8\pi}{7}, \dfrac{10\pi}{7}, \dfrac{12\pi}{7}$ are the roots of the equation

$7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta = 0$ --------------------------- (9.18)

Let $x = \sin\theta$

$\therefore$ (9.18) $\Rightarrow$ $7x - 56x^3 + 112x^5 - 64x^7 = 0$

(i.e) $64x^7 - 112x^5 + 56x^3 - 7x = 0$ ------------------------------------- (9.19)

Now $\sin\left(\dfrac{2\pi}{7}\right)$, $\sin\left(\dfrac{4\pi}{7}\right)$, $\sin\left(\dfrac{6\pi}{7}\right)$, $\sin\left(\dfrac{8\pi}{7}\right)$, $\sin\left(\dfrac{10\pi}{7}\right)$, $\sin\left(\dfrac{12\pi}{7}\right)$ are

the roots of $64x^6 - 112x^4 + 56x^2 - 7 = 0$ ----------------------------- (9.20).

Now $\sin\left(\dfrac{12\pi}{7}\right) = \sin\left(2\pi - \dfrac{2\pi}{7}\right) = -\sin\left(\dfrac{2\pi}{7}\right)$,

$\sin\left(\dfrac{10\pi}{7}\right) = \sin\left(2\pi - \dfrac{4\pi}{7}\right) = -\sin\left(\dfrac{4\pi}{7}\right)$,

$\sin\left(\dfrac{8\pi}{7}\right) = \sin\left(2\pi - \dfrac{6\pi}{7}\right) = -\sin\left(\dfrac{6\pi}{7}\right)$.

Thus $\pm\sin\left(\frac{2\pi}{7}\right)$, $\pm\sin\left(\frac{4\pi}{7}\right)$, $\pm\sin\left(\frac{6\pi}{7}\right)$ are the roots of (9.20).

Let $y = x^2$

Then (9.20) $\Rightarrow$ $64y^3 - 112y^2 + 56y - 7 = 0$ -------------------------- (9.21)

Now the roots of (9.21) are $\sin^2\left(\frac{2\pi}{7}\right)$, $\sin^2\left(\frac{4\pi}{7}\right)$, $\sin^2\left(\frac{6\pi}{7}\right)$.

Hence (9.21) is the required equation.

Example 9.2.6 :

Prove that $\sin\left(\frac{\pi}{5}\right)\cdot\sin\left(\frac{2\pi}{5}\right)\cdot\sin\left(\frac{3\pi}{5}\right)\cdot\sin\left(\frac{4\pi}{5}\right) = \frac{5}{16}$.

Proof :

Step 1 : First we shall find the expansion of $\sin 5\theta$.

We know that $\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$

$= \cos^5\theta + {}^5C_1\cos^{5-1}\theta(i\sin\theta) + {}^5C_2\cos^{5-2}\theta(i\sin\theta)^2$

$+ {}^5C_3\cos^{5-3}\theta(i\sin\theta)^3 + {}^5C_4\cos^{5-4}\theta(i\sin\theta)^4 + {}^5C_5\cos^{5-5}\theta(i\sin\theta)^5$

$= \cos^5\theta + i5\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - i10\cos^2\theta\sin^3\theta$

$+ 5\cos\theta\sin^4\theta + i\sin^5\theta$

Equating imaginary parts on both sides, we have,

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$

$= 5(1-2\sin^2\theta+\sin^4\theta)\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$

$= 5 + (-10-10)\sin^3\theta + (5+10+1)\sin^5\theta$

$= 5 - 20\sin^3\theta + 16\sin^5\theta$

If $\theta = \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$ then $\sin 5\theta = 0$

(i.e) $5 - 20\sin^3\theta + 16\sin^5\theta = 0$ -- (9.22)

Let $x = \sin\theta$

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$$\therefore (9.22) \Rightarrow 5-20x^3+16x^5=0 \quad\text{----------} (9.23)$$

Now (9.23) has roots $\sin\left(\frac{\pi}{5}\right), \sin\left(\frac{2\pi}{5}\right), \sin\left(\frac{3\pi}{5}\right), \sin\left(\frac{4\pi}{5}\right)$

Hence product of the roots $= \frac{5}{16}$

(i.e) $\sin\left(\frac{\pi}{5}\right)\cdot\sin\left(\frac{2\pi}{5}\right)\cdot\sin\left(\frac{3\pi}{5}\right)\cdot\sin\left(\frac{4\pi}{5}\right)=\frac{5}{16}$

This proves the problem.

Example 9.2.7 :

Prove that $\cos\left(\frac{2\pi}{7}\right), \cos\left(\frac{4\pi}{7}\right), \cos\left(\frac{6\pi}{7}\right)$ are the roots of the equation $8x^3+4x^2-4x-1=0$. Hence find the equation whose roots are

(i) $\sec\left(\frac{2\pi}{7}\right), \sec\left(\frac{4\pi}{7}\right), \sec\left(\frac{6\pi}{7}\right)$

(ii) $\sec^2\left(\frac{2\pi}{7}\right), \sec^2\left(\frac{4\pi}{7}\right), \sec^2\left(\frac{6\pi}{7}\right)$ and also prove that

$$\sec^2\left(\frac{2\pi}{7}\right)+\sec^2\left(\frac{4\pi}{7}\right)+\sec^2\left(\frac{6\pi}{7}\right)=24$$

(iii) $\tan^2\left(\frac{2\pi}{7}\right), \tan^2\left(\frac{4\pi}{7}\right), \tan^2\left(\frac{6\pi}{7}\right)$ and also prove that

$$\tan^2\left(\frac{2\pi}{7}\right)+\tan^2\left(\frac{4\pi}{7}\right)+\tan^2\left(\frac{6\pi}{7}\right)=21$$

Proof :

Let θ be one of the values $\frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}$

(i.e) $\theta=\frac{2n\pi}{7}$, $n=1,2,3$

(i.e) $7\theta=2\pi$, $n=1,2,3$

(i.e) $4\theta=2n\pi-3\theta$, $n=1,2,3$

$\therefore \cos 4\theta=\cos(2n\pi-3\theta)$

(i.e) $\cos 4\theta=\cos 3\theta$

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(i.e) $\cos^4\theta - {}^4C_2\cos^2\theta\sin^2\theta + {}^4C_4\sin^4\theta = 4\cos^3\theta - 3\cos\theta$

(i.e) $\cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2 - 4\cos^3\theta + 3\cos\theta = 0$

(i.e) $x^4 - 6x^2(1-x^2) + (1-x^2)^2 - 4x^3 + 3x = 0$ where $x = \cos\theta$

(i.e) $x^4 - 6x^2 + 6x^4 + 1 - 2x^2 + x^4 - 4x^3 + 3x = 0$

(i.e) $8x^4 - 4x^3 - 8x^2 + 3x + 1 = 0$

(i.e) $(x-1)(8x^3 + 4x^2 - 4x - 1) = 0$

(i.e) $x - 1 = 0$ or $8x^3 + 4x^2 - 4x - 1 = 0$

If $x - 1 = 0$ then $x = 1$

$\Rightarrow \cos\theta = 1$

$\Rightarrow \theta = 0$

$\therefore \cos\left(\frac{2\pi}{7}\right), \cos\left(\frac{4\pi}{7}\right), \cos\left(\frac{6\pi}{7}\right)$ are the roots of

$8x^3 + 4x^2 - 4x - 1 = 0$ -- (9.24)

Proof (i) : put $x = \frac{1}{y}$ in (9.24), we get, $8\left(\frac{1}{y}\right)^3 + 4\left(\frac{1}{y}\right)^2 - 4\left(\frac{1}{y}\right) - 1 = 0$

(i.e) $8 + 4y - 4y^2 - y^3 = 0$

(i.e) $y^3 + 4y^2 - 4y - 8 = 0$ --- (9.25)

Now the roots of (9.25) are $\sec\left(\frac{2\pi}{7}\right), \sec\left(\frac{4\pi}{7}\right), \sec\left(\frac{6\pi}{7}\right)$.

Thus (9.25) is the required equation whose roots are

$\sec\left(\frac{2\pi}{7}\right), \sec\left(\frac{4\pi}{7}\right), \sec\left(\frac{6\pi}{7}\right)$.

This proves (i).

Proof of (ii) : Let $z = \frac{1}{x^2}$

(i.e) $z = \frac{1}{\sqrt{x}}$

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Now put $z = \frac{1}{\sqrt{x}}$ in (9.24), we have, $8\left(\frac{1}{\sqrt{x}}\right)^3 + 4\left(\frac{1}{\sqrt{x}}\right)^2 - 4\left(\frac{1}{\sqrt{x}}\right) - 1 = 0$

(i.e) $8\left(\frac{1}{x\sqrt{x}}\right) + 4\left(\frac{1}{x}\right) - 4\left(\frac{1}{\sqrt{x}}\right) - 1 = 0$

(i.e) $8 + 4\sqrt{x} - 4x - x\sqrt{x} = 0$

(i.e) $8 - 4x = x\sqrt{x} - 4\sqrt{x}$

(i.e) $8 - 4x = \sqrt{x}(x - 4)$

(i.e) $(8 - 4x)^2 = x(x - 4)^2$

(i.e) $64 + 16x^2 - 64x = x(x^2 + 16 - 8x)$

(i.e) $64 + 16x^2 - 64x = x^3 + 16x - 8x^2$

(i.e) $x^3 + 16x - 8x^2 - 16x^2 + 64x - 64 = 0$

(i.e) $x^3 - 24x^2 + 80x - 64 = 0$ -- (9.26)

Now (9.26) has roots $\sec^2\left(\frac{2\pi}{7}\right)$, $\sec^2\left(\frac{4\pi}{7}\right)$, $\sec^2\left(\frac{6\pi}{7}\right)$

Again product of the roots of (9.26) = $-(-64)$

(i.e) $\sec^2\left(\frac{2\pi}{7}\right)\cdot\sec^2\left(\frac{4\pi}{7}\right)\cdot\sec^2\left(\frac{6\pi}{7}\right) = 64$

This proves (ii)

Proof of (iii) : Now $\tan^2\theta = \sec^2\theta - 1$

Thus the equation whose roots are $\tan^2\left(\frac{2\pi}{7}\right)$, $\tan^2\left(\frac{4\pi}{7}\right)$, $\tan^2\left(\frac{6\pi}{7}\right)$ is obtained by substituting $z = x - 1$ in (9.26)

(i.e) to substituting $x = z + 1$ in (9.26), we get,

$(z+1)^3 - 24(z+1)^2 + 80(z+1) - 64 = 0$

(i.e) $(z^3 + 3z^2 + 3z + 1) - 24(z^2 + 2z + 1) + 80(z+1) - 64 = 0$

(i.e) $z^3 + -21z^2 + 35z + 17 = 0$ -- (9.27)

Thus (9.27) has the roots $\tan^2\left(\frac{2\pi}{7}\right)$, $\tan^2\left(\frac{4\pi}{7}\right)$, $\tan^2\left(\frac{6\pi}{7}\right)$

And sum of the roots = $-(-21)$

(i.e) $\tan^2\left(\frac{2\pi}{7}\right)+\tan^2\left(\frac{4\pi}{7}\right)+\tan^2\left(\frac{6\pi}{7}\right)=21$

This proves (iii)

Example 9.2.8 :

Prove that the equation whose roots are $\tan^2\left(\frac{\pi}{11}\right)$, $\tan^2\left(\frac{2\pi}{11}\right)$, $\tan^2\left(\frac{3\pi}{11}\right)$,

$\tan^2\left(\frac{4\pi}{11}\right)$, $\tan^2\left(\frac{5\pi}{11}\right)$

is $x^5-55x^4+330x^3-462x^2+165x-11=0$. Hence deduce that

$\tan^2\left(\frac{\pi}{11}\right)\cdot\tan^2\left(\frac{2\pi}{11}\right)\cdot\tan^2\left(\frac{3\pi}{11}\right)\cdot\tan^2\left(\frac{4\pi}{11}\right)\cdot\tan^2\left(\frac{5\pi}{11}\right)=\sqrt{11}$.

Proof :

Let θ be one of the values

$\frac{\pi}{11},\frac{2\pi}{11},\frac{3\pi}{11},\frac{4\pi}{11},\frac{5\pi}{11},\frac{6\pi}{11},\frac{7\pi}{11},\frac{8\pi}{11},\frac{9\pi}{11},\frac{10\pi}{11}$

(i.e) $\theta=\frac{n\pi}{11},\quad n=0,1,2,\ldots,10$

(i.e) $11\theta=n\pi,\quad n=0,1,2,\ldots,10$

(i.e) $\tan(11\theta)=\tan(n\pi),\quad n=0,1,2,\ldots,10$

$\Rightarrow \tan(11\theta)=0$

$$\Rightarrow \frac{11\tan\theta-{}^{11}C_3\tan^3\theta+{}^{11}C_5\tan^5\theta-\ldots-{}^{11}C_{11}\tan^{11}\theta}{1-{}^{11}C_2\tan^2\theta+{}^{11}C_4\tan^4\theta-\ldots-{}^{11}C_{10}\tan^{10}\theta}=0$$

$\Rightarrow 11\tan\theta-{}^{11}C_3\tan^3\theta+{}^{11}C_5\tan^5\theta-\ldots-{}^{11}C_{11}\tan^{11}\theta=0$

$\Rightarrow 11x-165x^3+462x^5-330x^7+55x^9-x^{11}=0$ -------------------- (9.28)

where

$x=\tan\theta$.

(i.e) $x(11-165x^2+462x^4-330x^6+55x^8-x^{10})=0$

(i.e) $x=0$ or $x^{10}-55x^8+330x^6-462x^4+165x^2-11=0$

Space for Hints

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Hence $x^{10}-55x^8+330x^6-462x^4+165x^2-11=0$ ------------------ (9.29)

is the equation having the roots $\tan\frac{\pi}{11},\tan\frac{2\pi}{11},\tan\frac{3\pi}{11},\tan\frac{4\pi}{11},\tan\frac{5\pi}{11}$,

$\tan\frac{6\pi}{11},\tan\frac{7\pi}{11},\tan\frac{8\pi}{11},\tan\frac{9\pi}{11},\tan\frac{10\pi}{11}$.

Now $\tan\frac{10\pi}{11}=\tan\left(\pi-\frac{\pi}{11}\right)=-\tan\frac{\pi}{11}$,

$$\tan\frac{9\pi}{11}=\tan\left(\pi-\frac{2\pi}{11}\right)=-\tan\frac{2\pi}{11},$$

$$\tan\frac{8\pi}{11}=\tan\left(\pi-\frac{3\pi}{11}\right)=-\tan\frac{3\pi}{11},$$

$$\tan\frac{7\pi}{11}=\tan\left(\pi-\frac{4\pi}{11}\right)=-\tan\frac{4\pi}{11},$$

$$\tan\frac{6\pi}{11}=\tan\left(\pi-\frac{5\pi}{11}\right)=-\tan\frac{5\pi}{11}$$

$\therefore$ (9.29) has roots $\pm\tan\frac{\pi}{11},\pm\tan\frac{2\pi}{11},\pm\tan\frac{3\pi}{11},\pm\tan\frac{4\pi}{11},\pm\tan\frac{5\pi}{11}$

Let $y=x^2$

$\therefore$ (9.29) $\Rightarrow$ $y^5-55y^4+330y^3-462y^2+165y-11=0$ ------------ (9.30)

Here the equation (9.30) has roots

$$\tan^2\left(\frac{\pi}{11}\right),\tan^2\left(\frac{2\pi}{11}\right),\tan^2\left(\frac{3\pi}{11}\right),\tan^2\left(\frac{4\pi}{11}\right),\tan^2\left(\frac{5\pi}{11}\right).$$

Again product of the roots = $-(-11)$

(i.e) $\tan^2\left(\frac{\pi}{11}\right)\cdot\tan^2\left(\frac{2\pi}{11}\right)\cdot\tan^2\left(\frac{3\pi}{11}\right)\cdot\tan^2\left(\frac{4\pi}{11}\right)\cdot\tan^2\left(\frac{5\pi}{11}\right)=11$

(i.e) $\tan\left(\frac{\pi}{11}\right)\cdot\tan\left(\frac{2\pi}{11}\right)\cdot\tan\left(\frac{3\pi}{11}\right)\cdot\tan\left(\frac{4\pi}{11}\right)\cdot\tan\left(\frac{5\pi}{11}\right)=\sqrt{11}$

The negative sign is omitted because all terms of the expression on the left side are positive, each angle involved being acute.

Check your progress

Questions :

(1) Express $\dfrac{\sin 9\theta}{\sin\theta}$ as a polynomial in $\cos\theta$ and deduce that

(i) $\sec^2\left(\dfrac{\pi}{9}\right)+\sec^2\left(\dfrac{2\pi}{9}\right)+\sec^2\left(\dfrac{4\pi}{9}\right)=36$

(i) $\sec\left(\dfrac{\pi}{9}\right)\cdot\sec\left(\dfrac{2\pi}{9}\right)\cdot\sec\left(\dfrac{4\pi}{9}\right)=8$

(2) Prove that the roots of the equation $x^3-21x^2+35x-7=0$ are

$\tan^2\left(\dfrac{\pi}{7}\right),\tan^2\left(\dfrac{2\pi}{7}\right),\tan^2\left(\dfrac{3\pi}{7}\right)$.

9.3. Expansion of $\sin^n\theta$, $\cos^n\theta$, $\tan^n\theta$

Example 9.3.1 :

Find the expansion of $\cos^n\theta$ where n is positive integer.

Solution :

Let $x=\cos\theta+i\sin\theta$

$\therefore\ \dfrac{1}{x}=\cos\theta-i\sin\theta$

Now $x+\dfrac{1}{x}=2\cos\theta,\ \ x-\dfrac{1}{x}=2i\sin\theta$

Again $x^n=\cos n\theta+i\sin n\theta$

$\therefore\ \dfrac{1}{x^n}=\cos n\theta-i\sin n\theta$

Now $x^n+\dfrac{1}{x^n}=2\cos n\theta,\ \ x^n-\dfrac{1}{x^n}=2i\sin n\theta$

Now $(2\cos\theta)^n=\left(x+\dfrac{1}{x}\right)^n$

$$=x^n+{}^nC_1x^{n-1}\left(\frac{1}{x}\right)+{}^nC_2x^{n-2}\left(\frac{1}{x^2}\right)+\text{L}+{}^nC_{(n-1)}x\left(\frac{1}{x^{n-1}}\right)+\frac{1}{x^n}.$$

Space for Hints

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$$= \left(x^n + \frac{1}{x^n}\right) + {}^nC_1\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + {}^nC_2\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + L$$

(i.e) $2^n \cos^n\theta = 2\cos n\theta + {}^nC_1(2\cos(n-2)\theta) + {}^nC_2(2\cos(n-4)\theta) + L$

(i.e) $2^{n-1} \cos^n\theta = \cos n\theta + {}^nC_1 \cos(n-2)\theta + {}^nC_2 \cos(n-4)\theta + L$

Example 9.3.2 :

Find the expansion of $\sin^n\theta$ where n is positive integer.

Solution :

Let $x = \cos\theta + i\sin\theta$

$\therefore \ \frac{1}{x} = \cos\theta - i\sin\theta$

Now $x + \frac{1}{x} = 2\cos\theta, \ \ x - \frac{1}{x} = 2i\sin\theta$

Again $x^n = \cos n\theta + i\sin n\theta$

$\therefore \ \frac{1}{x^n} = \cos n\theta - i\sin n\theta$

Now $x^n + \frac{1}{x^n} = 2\cos n\theta, \ \ x^n - \frac{1}{x^n} = 2i\sin n\theta$

Now $(2i\sin\theta)^n = \left(x - \frac{1}{x}\right)^n$

$$= x^n - {}^nC_1 x^{n-1}\left(\frac{1}{x}\right) + {}^nC_2 x^{n-2}\left(\frac{1}{x^2}\right) - {}^nC_3 x^{n-3}\left(\frac{1}{x^3}\right) + L \quad \text{----} \ (9.31)$$

Case (1) : Let n be even.

If n is even, then the number of terms in the RHS of (9.31) is odd and the signs of the terms are alternatively positive and negative and the last term is positive.

Now $(2i\sin\theta)^n = \left(x - \frac{1}{x}\right)^n$

$$= x^n - {}^nC_1 x^{n-1}\left(\frac{1}{x}\right) + {}^nC_2 x^{n-2}\left(\frac{1}{x^2}\right) - L + (-1)^n\, {}^nC_{(n-1)} x\left(\frac{1}{x^{n-1}}\right) + \frac{1}{x^n}.$$

$$= \left(x^n + \frac{1}{x^n}\right) - {}^nC_1\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + {}^nC_2\left(x^{n-4} + \frac{1}{x^{n-4}}\right) - \mathrm{L}$$

(i.e) $2^n(-1)^{n/2}\sin^n\theta = 2\cos n\theta - {}^nC_1(2\cos(n-2)\theta) + {}^nC_2(2\cos(n-4)\theta) - \mathrm{L}$

(i.e) $2^{n-1}(-1)^{n/2}\sin^n\theta = \cos n\theta - {}^nC_1\cos(n-2)\theta + {}^nC_2\cos(n-4)\theta - \mathrm{L}$

Space for Hints

Case (2) : Let n be odd integer.

$$(2\,i\sin\theta)^n = \left(x - \frac{1}{x}\right)^n$$

$$= x^n - {}^nC_1 x^{n-1}\left(\frac{1}{x}\right) + {}^nC_2 x^{n-2}\left(\frac{1}{x^2}\right) + \mathrm{L}\ -\frac{1}{x^n}.$$

$$= \left(x^n - \frac{1}{x^n}\right) - {}^nC_1\left(x^{n-2} - \frac{1}{x^{n-2}}\right) + {}^nC_2\left(x^{n-4} - \frac{1}{x^{n-4}}\right) - \mathrm{L}$$

(i.e) $2^n\ i^n\sin^n\theta = 2\,i\sin n\theta - {}^nC_1(2i\sin(n-2)\theta) + {}^nC_2(2i\ \sin(n-4)\theta) - \mathrm{L}$

(i.e) $2^{n-1}(-1)^{\frac{n-1}{2}}\ \sin^n\theta = \sin n\theta - {}^nC_1\sin(n-2)\theta + {}^nC_2\sin(n-4)\theta - \mathrm{L}$

Example 9.3.3 :

Prove that $2^6\cos^7\theta = \cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta$

Solution :

Let $x = \cos\theta + i\sin\theta$

$\therefore\ \frac{1}{x} = \cos\theta - i\sin\theta$

Now $x + \frac{1}{x} = 2\cos\theta$, $x - \frac{1}{x} = 2i\sin\theta$

Again $x^n = \cos n\theta + i\sin n\theta$

$\therefore\ \frac{1}{x^n} = \cos n\theta - i\sin n\theta$

Now $x^n + \frac{1}{x^n} = 2\cos n\theta$, $x^n - \frac{1}{x^n} = 2i\sin n\theta$

Now $(2\cos\theta)^7 = \left(x + \frac{1}{x}\right)^7$

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$$= x^7 + {}^7C_1 x^6\left(\frac{1}{x}\right) + {}^7C_2 x^5\left(\frac{1}{x^2}\right) + {}^7C_3 x^4\left(\frac{1}{x^3}\right) + {}^7C_4 x^3\left(\frac{1}{x^4}\right)$$

$$+ {}^7C_5 x^2\left(\frac{1}{x^5}\right) + {}^7C_6 x^6\left(\frac{1}{x}\right) + {}^7C_7\left(\frac{1}{x^7}\right)$$

$$= \left(x^7 + \frac{1}{x^7}\right) + 7\left(x^5 + \frac{1}{x^5}\right) + 21\left(x^3 + \frac{1}{x^3}\right) + 35\left(x + \frac{1}{x}\right)$$

(i.e) $2^7 \cos^7\theta = 2\cos 7\theta + 7(2\cos 5\theta) + 21(2\cos 3\theta) + 35(2\cos\theta)$

(i.e) $2^6 \cos^7\theta = \cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta$

This proves the problem.

Example 9.3.4 :

Prove that $2^7 \sin^8\theta = \cos 8\theta - 8\cos 6\theta + 28\cos 4 - 56\cos 2\theta + 35$

Solution :

Let $x = \cos\theta + i\sin\theta$

$\therefore\ \frac{1}{x} = \cos\theta - i\sin\theta$

Now $x + \frac{1}{x} = 2\cos\theta$, $x - \frac{1}{x} = 2i\sin\theta$

Again $x^n = \cos n\theta + i\sin n\theta$

$\therefore\ \frac{1}{x^n} = \cos n\theta - i\sin n\theta$

Now $x'' + \frac{1}{x^n} = 2\cos n\theta$, $x^n - \frac{1}{x^n} = 2i\sin n\theta$

Now $(2\ i\ \sin)^8 = \left(x - \frac{1}{x}\right)^7$

$$= x^8 - {}^8C_1 x^7 \frac{1}{x} + {}^8C_2 x^6 \frac{1}{x^2} - {}^8C_3 x^5 \frac{1}{x^3} + {}^8C_4 x^4 \frac{1}{x^4} - {}^8C_5 x^3 \frac{1}{x^5}$$

$$+ {}^8C_6 x^2 \frac{1}{x^6} - {}^8C_7 x \frac{1}{x^7} + {}^8C_8 \frac{1}{x^8}$$

$$= \left(x^8 + \frac{1}{x^8}\right) - 8\left(x^6 + \frac{1}{x^6}\right) + 28\left(x^4 + \frac{1}{x^4}\right) - 56\left(x + \frac{1}{x}\right) + 70$$

$$= 2\cos 8\theta - 8(2\cos 6\theta) + 28(2\cos 4\theta) - 56(2\cos 2\theta) + 70$$

(i.e) $2^8 \sin^8\theta = 2\cos 8\theta - 8(2\cos 6\theta) + 28(2\cos 4\theta) - 56(2\cos 2\theta) + 70$

(i.e) $2^7 \sin^8\theta = \cos 8\theta - 8\cos 6\theta + 28\cos 4\theta) - 56\cos 2\theta) + 35$

This proves the problem.

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Example 9.3.5 :

Evaluate $\lim\limits_{x\to 0} \dfrac{\tan x - \sin x}{\sin^3 x}$

Solution :

Let L = $\lim\limits_{x\to 0} \dfrac{\tan x - \sin x}{\sin^3 x}$

We know that

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \text{L}$$

and $\sin\theta = \theta - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} + \text{L}$

and $\sin 3A = 3\sin A - 4\sin^3 A$

$$\Rightarrow\ 4\sin^3 A = 3\sin A - \sin 3A$$

$$\Rightarrow\ \sin^3 A = \frac{1}{4}[3\sin A - \sin 3A]$$

$$\Rightarrow\ \sin^3 x = \frac{1}{4}[3\sin x - \sin 3x]$$

$$\Rightarrow\ \sin^3 x = \frac{1}{4}\left\{3\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \text{L}\right] - \left[(3x) - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} + \text{L}\right]\right\}$$

$$\Rightarrow\ \sin^3 x = \frac{1}{4}x^3\left\{4 + \left(\frac{3-3^5}{5!}\right)x^2 + \text{L}\right\}$$

and $\tan x - \sin x = \left\{x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \text{L}\right\} - \left\{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \text{L}\right\}$

$$= x^3\left(\frac{1}{3} + \frac{1}{3!}\right) + x^5\left(\frac{2}{5} - \frac{1}{5!}\right) + \text{L}$$

$$= \frac{1}{2}x^3 + \left(\frac{2}{5} - \frac{1}{5!}\right)x^5 + \text{L}$$

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$$\therefore \lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x} = \lim_{x\to 0}\left[\frac{\frac{1}{2}x^3+\left(\frac{2}{5}-\frac{1}{5!}\right)x^5+\text{L}}{\frac{1}{4}x^3\left\{4+\left(\frac{3-3^5}{5!}\right)x^2+\text{L}\right\}}\right]$$

$$= \lim_{x\to 0}\left[\frac{x^3\left\{\frac{1}{2}+\left(\frac{2}{5}-\frac{1}{5!}\right)x^2+\text{L}\right\}}{\frac{1}{4}x^3\left\{4+\left(\frac{3-3^5}{5!}\right)x^2+\text{L}\right\}}\right]$$

$$= \lim_{x\to 0}\left[\frac{x^3\left\{\frac{1}{2}+\left(\frac{2}{5}-\frac{1}{5!}\right)x^2+\text{L}\right\}}{\frac{1}{4}x^3\left\{4+\left(\frac{3-3^5}{5!}\right)x^2+\text{L}\right\}}\right]$$

$$= \frac{1/2}{4/4} = \frac{1}{2}$$

Thus $\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x} = \frac{1}{2}$

Example 9.3.6 :

Evaluate $\lim_{x\to 0}\frac{\sin 2x-2\sin x}{x^3}$

Solution :

Let L = $\lim_{x\to 0}\frac{\sin 2x-2\sin x}{x^3}$

We know that, $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}+\text{L}$

and $\sin 2x = 2x-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}+\text{L}$

$$\therefore \sin 2x-2\sin x = \left[2x-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}+\text{L}\right]-2\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}+\text{L}\right]$$

$$= \left[-\frac{2^3}{6}+\frac{2}{6}\right]x^3+\left[\frac{2^5}{5!}-\frac{2}{5!}\right]x^5+\text{L}$$

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Thus $\lim\limits_{x\to 0}\dfrac{\sin 2x - 2\sin x}{x^3} = \lim\limits_{x\to 0}\left\{\dfrac{\left[-\dfrac{2^3}{6}+\dfrac{2}{6}\right]x^3+\left[\dfrac{2^5}{5!}-\dfrac{2}{5!}\right]x^5+\mathrm{L}}{x^3}\right\}$

Thus $\lim\limits_{x\to 0}\dfrac{\sin 2x - 2\sin x}{x^3} = \lim\limits_{x\to 0}\left\{[-1]+\left[\dfrac{2^5}{5!}-\dfrac{2}{5!}\right]x^2+\mathrm{L}\right\}$

(i.e) $\lim\limits_{x\to 0}\dfrac{\sin 2x - 2\sin x}{x^3} = -1.$

Example 9.3.7 :

Evaluate $\lim\limits_{x\to \pi/2}\dfrac{\cos\theta - \sin 2\theta}{\cos 3\theta}$

Solution :

Let L = $\lim\limits_{x\to \pi/2}\dfrac{\cos\theta - \sin 2\theta}{\cos 3\theta}$

Let $x = \theta - \dfrac{\pi}{2}$

As $\theta \to \dfrac{\pi}{2}$ then $x \to 0$.

$$\therefore \cos\theta = \cos\left(x+\frac{\pi}{2}\right)$$

$$= -\sin x$$

$$= -\left[x - \frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\mathrm{L}\right]$$

and $\sin 2\theta = \sin(\pi + 2x)$

$$= -\sin 2x$$

$$= -\left[(2x) - \frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}-\frac{(2x)^7}{7!}+\mathrm{L}\right]$$

$$= -2x + \frac{(2x)^3}{3!}-\frac{(2x)^5}{5!}+\frac{(2x)^7}{7!}-\mathrm{L}$$

and $\cos 3\theta = \cos\left(\dfrac{3\pi}{2}+3x\right)$

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$$= \sin 3x$$

$$= (3x) - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!} + L$$

$$\therefore\ L = \lim_{x\to\pi/2} \frac{\cos\theta - \sin 2\theta}{\cos 3\theta}$$

$$= \lim_{x\to\pi/2} \frac{-\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + L\right] - \left[-2x + \frac{(2x)^3}{3!} - \frac{(2x)^5}{5!} + \frac{(2x)^7}{7!} - L\right]}{\left[(3x) - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!} + L\right]}$$

$$= \lim_{x\to\pi/2} \frac{x\left[1 + \frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - L + \frac{2^3x^2}{3!} - \frac{2^5x^4}{5!} + \frac{2^7x^6}{7!} - L\right]}{3x\left[1 - \frac{(3x)^2}{3!} + \frac{(3x)^4}{5!} - \frac{(3x)^6}{7!} + L\right]}$$

$$= \frac{1}{3}$$

Hence $\lim_{x\to\pi/2} \frac{\cos\theta - \sin 2\theta}{\cos 3\theta} = \frac{1}{3}$.

Check your progress

Questions :

(1) Evaluate $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$

(2) Evaluate $\lim_{x\to\pi/2} \frac{\sin x - \cos 2x}{\cos^2 x}$

Example 9.3.8 :

Given that $\frac{\sin\theta}{\theta} = \frac{2165}{2166}$, show that θ is nearly the circular measure of 3°.

Solution :

Now $\frac{\sin\theta}{\theta} = \frac{2165}{2166}$

(i.e) $\frac{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - L}{\theta} = \frac{2165}{2166}$

(i.e) $1-\dfrac{\theta^2}{3!}+\dfrac{\theta^4}{5!}=\dfrac{2165}{2166}$ approximately.

(i.e) $\dfrac{\theta^2}{3!}-\dfrac{\theta^4}{5!}=1-\dfrac{2165}{2166}$

(i.e) $\dfrac{\theta^2}{6}-\dfrac{\theta^4}{120}=\dfrac{1}{2166}$

For the first-degree approximations, we have, $\dfrac{\theta^2}{6}=\dfrac{1}{2166}$

(i.e) $\theta^2=\dfrac{6}{2166}$

(i.e) $\theta=\dfrac{1}{19}$ of a radian.

(i.e) $\theta=3^{\circ}1'$

(i.e) $\theta=3^{\circ}$ approximately.

This proves the problem.

Example 9.3.9 :

Given that $\dfrac{\tan\theta}{\theta}=\dfrac{2524}{2523}$, show that the value of θ is $1^{\circ}58'$.

Solution : Now $\dfrac{\tan\theta}{\theta}=\dfrac{2524}{2523}$

(i.e) $\dfrac{1+\dfrac{\theta^3}{3}+\dfrac{2\theta^5}{15}-L}{\theta}=\dfrac{2524}{2523}$

(i.e) $1+\dfrac{\theta^2}{3}+\dfrac{2\theta^4}{5}=\dfrac{2524}{2523}$ approximately.

(i.e) $\dfrac{\theta^2}{3}+\dfrac{2\theta^4}{5}=\dfrac{2524}{2523}-1$

(i.e) $\dfrac{\theta^2}{3}+\dfrac{2\theta^4}{5}=\dfrac{1}{2523}$

For the first degree approximations, we have, $\dfrac{\theta^2}{3}=\dfrac{1}{2523}$

(i.e) $\theta^2=\dfrac{3}{2523}=\dfrac{1}{841}$ of a radian.

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(i.e) $\theta = \frac{1}{29}$ of a radian.

(i.e) $\theta = 1^\circ 58'$ approximately

This proves the problem.

Check your progress

Question

Given that $\frac{\sin\theta}{\theta} = \frac{5045}{5046}$, show that the value of θ is $1^\circ 58'$.

Summary

In this unit we have learned that the method of finding roots of biquadratic equation by Ferrari's method, expansion of $\sin n\theta$, $\cos n\theta$, $\tan n\theta$, $\sin^n\theta$, $\cos^n\theta$, $\tan^n\theta$, $\sin^m\theta\cos^n\theta$.

Further Reading

You can also refer the following books for further reading.

(1) Trigonometry by T.K.Manicavachagom Pillai and others

(2) Trigonometry by Arumugam and others.

UNIT X

Hyperbolic Functions and Gregory's Series

Space for Hints

Unit Objectives

Unit Structure

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Objectives :

In this unit, we are going to discuss Hyperbolic functions, Inverse hyperbolic functions, Logarithmic of complex numbers and Gregory series

After completing this unit, students may able to know

- o Hyperbolic functions
- o Inverse hyperbolic functions
- o Logarithmic of complex numbers
- o Gregory series

10.1. Hyperbolic functions

Definition : We define hyperbolic functions as

$$\sinh x = \frac{e^x - e^{-x}}{2},$$

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

$$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}},$$

$$\cos ech\, x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}},$$

$$\sec h\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Note :

We know that $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \text{L}$ -------------------------(10.1)

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \text{L} \quad \text{------------------} \quad (10.2)$$

Now (10.1) + (10.2) $\Rightarrow$ $e^x + e^{-x} = 2\left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \text{L}\right]$

(i.e) $\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \text{L}$

(i.e) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \text{L}$ ------------------ (10.3)

Now (10.1) – (10.2) $\Rightarrow$ $e^x - e^{-x} = 2\left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \text{L}\right]$

(i.e) $\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \text{L}$

(i.e) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \text{L}$ ------------------ (10.4)

Further $\cosh(0) = 1$ and $\sinh(0) = 0$

Example 10.1.1 :

Prove that $\cosh^2 x - \sinh^2 x = 1$

Proof :

LHS $= \cosh^2 x - \sinh^2 x$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{1}{4}\left[\left(e^{2x} + e^{-2x} + 2\right) - \left(e^{2x} + e^{-2x} - 2\right)\right]$$

$$= \frac{1}{4}(4)$$

$$= 1$$

$$= \text{RHS}$$

Hence $\cosh^2 x - \sinh^2 x = 1$

Example 10.1.2 :

Prove that $\sinh 2x = 2\sinh x \cosh x$

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Proof :

RHS $= 2\sinh x \cosh x$

$$= 2\left(\frac{e^{x}-e^{-x}}{2}\right)\left(\frac{e^{x}+e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left(e^{2x}-e^{-2x}\right)$$

$$= \sinh 2x$$

$$= \text{RHS}$$

$\therefore\ \sinh 2x = 2\sinh x \cosh x$.

Example 10.1.3 :

Prove that $\cosh^2 x + \sinh^2 x = \cosh 2x$

Proof :

LHS $= \cosh^2 x + \sinh^2 x$

$$= \left(\frac{e^{x}+e^{-x}}{2}\right)^2 + \left(\frac{e^{x}-e^{-x}}{2}\right)^2$$

$$= \frac{1}{4}\left[e^{2x}+e^{-2x}+2+e^{2x}+e^{-2x}-2\right]$$

$$= \frac{1}{2}\left[e^{2x}+e^{-2x}\right]$$

$$= \cosh 2x$$

Thus $\cosh^2 x + \sinh^2 x = \cosh 2x$.

Check your progress

Question

(1) Prove that $\cosh 2x = 2\cosh^2 x - 1$

(2) Prove that $\cosh 2x = 1 + 2\sinh^2 x$

Example 10.1.4 :

Prove the following :

(1) $\sin(ix) = i \sinh x$

(2) $\cos(ix) = \cosh x$

(3) $\tan(ix) = i \tanh x$

Proof of (1) :

We know that $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + L$

$$\therefore \sin(ix) = (ix) - \frac{(ix)^3}{3!} + \frac{(ix)^5}{5!} - \frac{(ix)^7}{7!} + L$$

$$= ix + i\frac{x^3}{3!} + i\frac{x^5}{5!} - i\frac{x^7}{7!} + L$$

$$= i\left[x + \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + L\right]$$

$$= i \sinh x$$

Hence $\sin(ix) = i \sinh x$

Proof of (2) :

We know that $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + L$

$$\therefore \cos(ix) = 1 - \frac{(ix)^2}{2!} + \frac{(ix)^4}{4!} - \frac{(ix)^6}{6!} + L$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + L$$

$$= \cosh x$$

Hence $\cos(ix) = \cosh x$

Proof of (3) :

$$\tan(ix) = \frac{\sin(ix)}{\cos(ix)}.$$

$$= \frac{i \sinh x}{\cosh x}$$

$$= i \tanh x$$

Hence $\tan(ix) = i \tanh x$

Further, note that $\sinh x = -i\sin(ix)$,

$\cosh x = \cos(ix)$,

$\tanh x = -i\tan(ix)$

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Example 10.1.5 :

Prove that $\cosh 2x = 2\cosh^2 x - 1 = 1 + \sinh^2 x = \dfrac{1+\tanh^2 x}{1-\tanh^2 x}$

Proof :

Now $2\cosh^2 x - 1$

$$= 2\left(\frac{e^x + e^{-x}}{2}\right)^2 - 1$$

$$= 2 \cdot \frac{1}{4}\left(e^{2x} + e^{-2x} + 2 - 2\right)$$

$$= \frac{e^{2x} + e^{-2x}}{2}$$

$$= \cosh 2x$$

Thus $\cosh 2x = 2\cosh^2 x - 1$ --(10.5)

Again $1 + \sinh^2 x$

$$= 1 + 2\left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= 1 + 2 \cdot \frac{1}{4}\left(e^{2x} + e^{-2x}\right) - 1$$

$$= \frac{e^{2x} + e^{-2x}}{2}$$

$$= \cosh 2x$$

Thus $\cosh 2x = 1 + 2\sinh^2 x$ -- (10.6)

Again $\dfrac{1+\tanh^2 x}{1-\tanh^2 x} = \dfrac{\cosh^2 x + \sinh^2 x}{\cosh^2 x - \sinh^2 x}$

$$= \frac{\cosh 2x}{1}$$

$$= \cosh 2x$$

Thus $\dfrac{1+\tanh^2 x}{1-\tanh^2 x} = \cosh 2x$ -- (10.7)

Hence, from (10.5), (10.6) and (10.7), we have,

$$\cosh 2x = 2\cosh^2 x - 1 = 1 + \sinh^2 x = \frac{1+\tanh^2 x}{1-\tanh^2 x}.$$

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Example 10.1.6 :

Prove that $\sinh 3x = 3\sinh x + 4\sinh^3 x$

Proof :

$$\begin{aligned}
\text{LHS} &= \sinh 3x \\
&= \frac{1}{i}\sin(i3x) \\
&= \frac{1}{i}[3\sin(ix) - 4\sin^3(ix)] \\
&= \frac{1}{i}[3i\sinh x + 4i\sinh^3 x] \\
&= 3\sinh x + 4\sinh^3 x
\end{aligned}$$

Hence $\sinh 3x = 3\sinh x + 4\sinh^3 x$.

This proves the problem.

10.2. Inverse hyperbolic functions

We can express $\sinh^{-1} x$, $\cosh^{-1} x$, $\tanh^{-1} x$ in terms of logarithmic functions.

Example 10.2.1 :

Prove that $\sinh^{-1} x = \log_e\left(x + \sqrt{x^2+1}\right)$

Solution : Let $y = \sinh^{-1} x$

Now $y = \sinh^{-1} x$

$\Rightarrow \sinh y = x$

$$\Rightarrow \frac{e^y - e^{-y}}{2} = x$$

$$\Rightarrow e^y - \frac{1}{e^y} = 2x$$

$$\Rightarrow e^{2y} - 1 = 2xe^y$$

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$$\Rightarrow (e^y)^2 - 2xe^y - 1 = 0$$

$$\Rightarrow e^y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

$$\Rightarrow e^y = x \pm \sqrt{x^2 + 1}$$

$$\Rightarrow e^y = x + \sqrt{x^2 + 1} \quad (\because \ e^y > 0 \text{ always})$$

$$\Rightarrow y = \log_e \left(x + \sqrt{x^2 + 1} \right)$$

$$\Rightarrow \sinh^{-1} x = \log_e \left(x + \sqrt{x^2 + 1} \right)$$

This proves the problem

Example 10.2.2 :

Prove that $\cosh^{-1} x = \log_e \left(x + \sqrt{x^2 - 1} \right)$

Solution : Let $y = \cosh^{-1} x$

Now $y = \sinh^{-1} x$

$$\Rightarrow \cosh y = x$$

$$\Rightarrow \frac{e^y + e^{-y}}{2} = x$$

$$\Rightarrow e^y + \frac{1}{e^y} = 2x$$

$$\Rightarrow e^{2y} + 1 = 2xe^y$$

$$\Rightarrow (e^y)^2 - 2xe^y + 1 = 0$$

$$\Rightarrow e^y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$$

$$\Rightarrow e^y = x \pm \sqrt{x^2 - 1}$$

$\Rightarrow e^y = x+\sqrt{x^2-1}$ or $e^y = x-\sqrt{x^2-1}$

Now $e^y = x-\sqrt{x^2-1}$

$$\Rightarrow e^y = \left(x-\sqrt{x^2-1}\right)\cdot\frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$$

$$\Rightarrow e^y = \frac{x^2-x^2+1}{x+\sqrt{x^2-1}}.$$

$$\Rightarrow e^y = \frac{1}{x+\sqrt{x^2-1}}$$

$$\Rightarrow e^y = \left(x+\sqrt{x^2-1}\right)^{-1}$$

Thus $e^y = x+\sqrt{x^2-1}$ or $e^y = \left(x+\sqrt{x^2-1}\right)^{-1}$

$$\Rightarrow y = \log_e\left(x+\sqrt{x^2-1}\right) \text{ or } y = -\log_e\left(x+\sqrt{x^2-1}\right)$$

$$\Rightarrow y = \pm\log_e\left(x+\sqrt{x^2-1}\right)$$

Hence the principal value of $\cosh^{-1} x$ is $\log_e\left(x+\sqrt{x^2-1}\right)$

(i.e) $\cosh^{-1} x = \log_e\left(x+\sqrt{x^2-1}\right)$

This proves the problem.

Example 10.2.3 :

Prove that $\tanh^{-1} x = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$

Solution : Let $y = \tanh^{-1} x$

Now $y = \tanh^{-1} x$

$\Rightarrow \tanh y = x$

$$\Rightarrow \frac{e^y+e^{-y}}{e^y-e^{-y}} = x$$

$\Rightarrow e^y+e^{-y} = x(e^y-e^{-y})$

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$$\Rightarrow e^{y}+e^{-y}=xe^{y}-xe^{-y}$$

$$\Rightarrow e^{y}(1-x)=e^{-y}(1+x)$$

$$\Rightarrow \frac{e^{y}}{e^{-y}}=\log\left(\frac{1+x}{1-x}\right)$$

$$\Rightarrow e^{2y}=\frac{1+x}{1-x}$$

$$\Rightarrow e^{2y}=\log\left(\frac{1+x}{1-x}\right)$$

$$\Rightarrow 2y=\log\left(\frac{1+x}{1-x}\right)$$

$$\Rightarrow y=\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

This proves the problem.

Example 10.2.4 :

Prove that if $\tan(x+iy)=A+iB$, then $\dfrac{A}{B}=\dfrac{\sin 2x}{\sinh 2y}$

Solution : Given that $\tan(x+iy)=A+iB$

(i.e) $A+iB=\tan(x+iy)$

$$=\frac{\sin(x+iy)}{\cos(x+iy)}$$

$$=\frac{\sin(x+iy)}{\cos(x+iy)}\cdot\frac{2\cos(x-iy)}{2\cos(x-iy)}$$

$$=\frac{\sin(2x)+\sin(2iy)}{\cos(2x)+\cos(2iy)}$$

$$=\frac{\sin(2x)+i\sinh 2y)}{\cos(2x)+\cosh(2y)}$$

Equating real and imaginary parts on both sides, we have,

$$A=\frac{\sin(2x)}{\cos(2x)+\cosh(2y)}$$

and $B=\dfrac{\sinh(2y)}{\cos(2x)+\cosh(2y)}$

$$\therefore\ \frac{A}{B}=\frac{\dfrac{\sin(2x)}{\cos(2x)+\cosh(2y)}}{\dfrac{\sinh(2y)}{\cos(2x)+\cosh(2y)}}$$

(i.e) $\dfrac{A}{B}=\dfrac{\sin 2x}{\sinh 2y}$

This proves the problem.

Example 10.2.5 :

If $x+i\,y=\tan(A+i\,B)$, then prove that $x^2+y^2+2x\cot 2A=1$

Solution : Given that $x+i\,y=\tan(A+i\,B)$

Thus $x-i\,y=\tan(A-i\,B)$

$$\text{now } \cot 2A=\frac{1}{\tan 2A}$$

$$=\frac{1}{\tan[(A+iB)+(A-iB)]}$$

$$=\frac{1}{\dfrac{\tan(A+iB)+\tan(A-iB)}{1-\tan(A+iB)\cdot\tan(A-iB)}}$$

$$=\frac{1-\tan(A+iB)\cdot\tan(A-iB)}{\tan(A+iB)+\tan(A-iB)}$$

$$=\frac{1-(x+i\,y)\cdot(x-i\,y)}{(x+i\,y)+(x-i\,y)}$$

$$=\frac{1-(x^2+y^2)}{2x}$$

(i.e) $2x\cot 2A=1-(x^2+y^2)$

(i.e) $x^2+y^2+2x\cot 2A=1$

This proves the problem.

Example 10.2.6 :

If $\sin(\theta+i\varphi)=\tan\alpha+i\sec\alpha$, then prove that $\cot 2\theta\cosh 2\varphi=3$

Solution : Given that $\sin(\theta+i\varphi)=\tan\alpha+i\sec\alpha$

(i.e) $\sin(\theta+i\varphi)=\tan\alpha+i\sec\alpha$

Space for Hints

Space for Hints

(i.e) $\sin\theta\cos(i\varphi)+\cos\theta\sin(i\varphi)=\tan\alpha+i\sec\alpha$

(i.e) $\sin\theta\cosh\varphi+i\ \cos\theta\sinh\varphi=\tan\alpha+i\sec\alpha$

Equating real and imaginary parts on both sides, we have,

$\sin\theta\cosh\varphi=\tan\alpha$ and $\cos\theta\sinh\varphi=\sec\alpha$

(i.e) $\tan\alpha=\sin\theta\cosh\varphi$ and $\sec\alpha=\cos\theta\sinh\varphi$

We know that $\sec^2\alpha=1+\tan^2\alpha$

$\therefore\ (\cos\theta\sinh\varphi)^2=1+(\sin\theta\cosh\varphi)^2$

(i.e) $\cos^2\theta\sinh^2\varphi=1+\sin^2\theta\cosh^2\varphi$

(i.e) $\cos^2\theta\sinh^2\varphi=1+(1-\cos^2\theta)\cosh^2\varphi$

(i.e) $\cos^2\theta\sinh^2\varphi=1+\cosh^2\varphi-\cos^2\theta\cosh^2\varphi$

(i.e) $\cos^2\theta(\sinh^2\varphi+\cosh^2\varphi)=1+\cosh^2\varphi$

(i.e) $\cos^2\theta\cosh 2\varphi=1+\cosh^2\varphi$

(i.e) $\frac{1}{2}(1+\cos 2\theta)\cosh 2\varphi=1+\frac{1}{2}(1+\cosh 2\varphi)$

(i.e) $(1+\cos 2\theta)\cosh 2\varphi=2+(1+\cosh 2\varphi)$

(i.e) $\cosh 2\varphi+\cos 2\theta\cosh 2\varphi=3+\cosh 2\varphi$

(i.e) $\cos 2\theta\cosh 2\varphi=3$

This proves the problem

Example 10.2.7 :

If $u=\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$, then prove that $\cosh u=\sec\theta$.

Solution : Given that $u=\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

(i.e) $e^u=\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

$$=\frac{\tan\frac{\pi}{4}+\tan\frac{\theta}{2}}{1-\tan\frac{\pi}{4}\cdot\tan\frac{\theta}{2}}$$

$$= \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}$$

and $e^{-u} = \frac{1}{e^{u}}$

$$= \frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}$$

Thus $e^{u} + e^{-u} = \dfrac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}} + \dfrac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}$

$$= \frac{\left(1+\tan\frac{\theta}{2}\right)^2 + \left(1-\tan\frac{\theta}{2}\right)^2}{\left(1+\tan\frac{\theta}{2}\right)^2 \cdot \left(1-\tan\frac{\theta}{2}\right)^2}$$

$$= \frac{1+2\tan\frac{\theta}{2}+\tan^2\frac{\theta}{2}+1-2\tan\frac{\theta}{2}+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$

$$= \frac{2+2\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$

$$= \frac{2\left(1+\tan^2\frac{\theta}{2}\right)}{1-\tan^2\frac{\theta}{2}}$$

$$= 2\sec\theta$$

$\therefore \dfrac{e^{u}+e^{-u}}{2} = \sec\theta$

(i.e) $\cosh u = \sec\theta$

This proves the problem.

Example 10.2.7 :

Expand $\sinh^8\theta$ in a series of hyperbolic cosines of multiples of θ.

Space for Hints

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Solution :

$$\sinh^8\theta = \left(\frac{e^x - e^{-x}}{2}\right)^8$$

$$= \frac{1}{2^8}\Big[e^{8\theta} - {}^8C_1 e^{7\theta}e^{-\theta} + {}^8C_2 e^{6\theta}e^{-2\theta} - {}^8C_3 e^{5\theta}e^{-3\theta} + {}^8C_4 e^{4\theta}e^{-4\theta}$$

$$-{}^8C_5 e^{3\theta}e^{-5\theta} + {}^8C_6 e^{2\theta}e^{-6\theta} - {}^8C_7 e^{\theta}e^{-7\theta} + {}^8C_8 e^{-8\theta}\Big]$$

$$= \frac{1}{2^8}\left[\left(e^{8\theta}+\frac{1}{e^{8\theta}}\right) - 8\left(e^{6\theta}+\frac{1}{e^{6\theta}}\right) + 28\left(e^{4\theta}+\frac{1}{e^{4\theta}}\right) - 56\left(e^{2\theta}\right) + 70\right]$$

$$= \frac{1}{2^8}\left[\left(e^{8\theta}+\frac{1}{e^{8\theta}}\right) - 8\left(e^{6\theta}+\frac{1}{e^{6\theta}}\right) + 28\left(e^{4\theta}+\frac{1}{e^{4\theta}}\right) - 56\left(e^{2\theta}+\frac{1}{e^{2\theta}}\right) + 70\right]$$

$$= \frac{1}{2^7}(\cosh 8\theta - \cosh 6\theta + 28\cosh 4\theta - 56\cosh 2\theta + 70)$$

Thus $\sinh^8\theta = \frac{1}{2^7}(\cosh 8\theta - \cosh 6\theta + 28\cosh 4\theta - 56\cosh 2\theta + 70)$

Example 10.2.8 :

If $\cos\alpha + i\sin\alpha = \cos(\theta + i\varphi)$ prove that $\sin^2\theta = \pm\sin\alpha$

Proof :

Given that $\cos\alpha + i\sin\alpha = \cos(\theta + i\varphi)$

$$= \cos\theta\cos(i\varphi) - \sin\theta\sin(i\varphi)$$

$$= \cos\theta\cosh\varphi - i\sin\theta\sinh\varphi$$

Equating real and imaginary parts ob both sides, we have,

$\cos\alpha = \cos\theta\cosh\varphi$ --(10.8) and

$\sin\alpha = -\sin\theta\sinh\varphi$ --(10.9)

Now (10.8) $\Rightarrow$ $\cosh\varphi = \dfrac{\cos\alpha}{\cos\theta}$

and (10.9) $\Rightarrow$ $\sinh\varphi = -\dfrac{\sin\alpha}{\sin\theta}$

we know that, $\cosh^2\varphi - \sinh^2\varphi = 1$

$$\Rightarrow \frac{\cos^2\alpha}{\cos^2\theta} - \frac{\sin^2\alpha}{\sin^2\theta} = 1$$

$$\Rightarrow \frac{\cos^2\alpha \cdot \sin^2\theta - \sin^2\alpha \cdot \cos^2\theta}{\cos^2\theta \cdot \sin^2\theta} = 1$$

$$\Rightarrow \cos^2\alpha \cdot \sin^2\theta - \sin^2\alpha \cdot \cos^2\theta = \cos^2\theta \cdot \sin^2\theta$$

$$\Rightarrow (1-\sin^2\alpha)\cdot \sin^2\theta - \sin^2\alpha \cdot (1-\sin^2\theta) = (1-\sin^2\theta)\cdot \sin^2\theta$$

$$\Rightarrow \sin^2\theta - \sin^2\alpha \cdot \sin^2\theta - \sin^2\alpha + \sin^2\alpha \cdot \sin^2\theta = \sin^2\theta - \sin^4\theta$$

$$\Rightarrow -\sin^2\alpha = -\sin^4\theta$$

$$\Rightarrow \sin^4\theta = \sin^2\alpha$$

$$\Rightarrow \sin^2\theta = \pm \sin\alpha$$

This proves the problem.

Example 10.2.9 :

Separate into real and imaginary parts of $\tanh(1+i)$.

Solution :

$\tanh(1+i) = -i\tan(ix)$ (since $\tanh x = -i\tan(ix)$)

Now $i\tanh(1+i) = \tan(i-1)$

$$= \frac{\sin(i-1)}{\cos(i-1)}$$

$$= \frac{\sin(i-1)}{\cos(i-1)} \cdot \frac{2\cos(i+1)}{2\cos(i+1)}$$

$$= \frac{\sin(2i)-\sin(2)}{\cos(2i)+\cos(2)}$$

$$= \frac{i\sinh(2)-\sin(2)}{\cosh(2)+\cos(2)}$$

$$\therefore \tanh(1+i) = \frac{i}{i}\left[\frac{\sinh(2)+i\sin(2)}{\cosh(2)+\cos(2)}\right]$$

$$\text{(i.e) } \tanh(1+i) = \frac{\sinh(2)}{\cosh(2)+\cos(2)} + i\frac{\sin(2)}{\cosh(2)+\cos(2)}$$

Example 10.2.10 :

Find the real and imaginary parts of $\tan^{-1}(x+iy)$.

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Solution :

Let $a+ib=\tan^{-1}(x+iy)$

(i.e) $\tan(a+ib)=x+iy$

$\therefore \tan(a-ib)=x-iy$

Now $\tan 2a = \tan[(a+ib)+(a-ib)]$

$$= \frac{\tan(a+ib)+\tan(a-ib)}{1-\tan(a+ib)\cdot\tan(a-ib)}$$

$$= \frac{x+iy+x-iy}{1-(x+iy)(x-iy)}$$

$$= \frac{2x}{1-(x^2+y^2)}$$

Thus $\tan 2a = \dfrac{2x}{1-(x^2+y^2)}$

$\Rightarrow\ 2a=\tan^{-1}\left(\dfrac{2x}{1-x^2-y^2}\right)$

$\Rightarrow\ a=\dfrac{1}{2}\tan^{-1}\left(\dfrac{2x}{1-x^2-y^2}\right)$

Now $\tan 2ib = \tan[(a+ib)-(a-ib)]$

$$= \frac{\tan(a+ib)-\tan(a-ib)}{1+\tan(a+ib)\cdot\tan(a-ib)}$$

$$= \frac{x+iy-x+iy}{1+(x+iy)(x-iy)}$$

$$= \frac{2iy}{1+(x^2+y^2)}$$

Thus $\tan 2ib = \dfrac{2iy}{1+x^2+y^2}$

$\Rightarrow\ i\tanh(2b)=\dfrac{2iy}{1+x^2+y^2}$

$\Rightarrow\ \tanh(2b)=\dfrac{2y}{1+x^2+y^2}$

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$$\Rightarrow \ 2b = \tanh^{-1}\left(\frac{2y}{1+x^2+y^2}\right)$$

$$\Rightarrow \ b = \frac{1}{2}\tanh^{-1}\left(\frac{2y}{1+x^2+y^2}\right)$$

Example 10.2.11 :

If $\tan(\theta+i\varphi) = \cos\alpha + i\sin\alpha$ prove that (i) $\theta = \frac{n\pi}{2}+\frac{\pi}{4}$ and

(ii) $\varphi = \frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$.

Solution : Given that $\tan(\theta+i\varphi) = \cos\alpha + i\sin\alpha$

$\therefore \tan(\theta-i\varphi) = \cos\alpha - i\sin\alpha$

Now $\tan(\theta+i\varphi)+\tan(\theta-i\varphi)$

$$= (\cos\alpha + i\sin\alpha)+(\cos\alpha - i\sin\alpha)$$

$$= 2\cos\alpha$$

and $\tan(\theta+i\varphi)+\tan(\theta-i\varphi)$

$$= (\cos\alpha + i\sin\alpha)-(\cos\alpha - i\sin\alpha)$$

$$= \cos\alpha + i\sin\alpha - \cos\alpha + i\sin\alpha)$$

$$= 2i\sin\alpha$$

and $\tan(\theta+i\varphi)\cdot\tan(\theta-i\varphi)$

$$= (\cos\alpha + i\sin\alpha)\cdot(\cos\alpha - i\sin\alpha)$$

$$= \cos^2\alpha + \sin^2\alpha$$

$$= 1$$

Now $\tan 2\theta \ = \ \tan[(\theta+i\varphi)+(\theta-i\varphi)]$

$$= \frac{\tan(\theta+i\varphi)+\tan(\theta-i\varphi)}{1-\tan(\theta+i\varphi)\cdot\tan(\theta-i\varphi)}$$

$$= \frac{2\cos\alpha}{1-1}$$

$$= \infty$$

(i.e.) $2\theta \ = \ \tan^{-1}(\infty)$

$$= n\pi + \frac{\pi}{2}$$

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$\therefore\ \theta = \frac{n\pi}{2} + \frac{\pi}{4}$

This proves (i).

Again $\tan(2i\varphi) = \tan[(\theta + i\varphi) - (\theta - i\varphi)]$

(i.e.) $i\tanh(2\varphi) = \dfrac{\tan(\theta + i\varphi) - \tan(\theta - i\varphi)}{1 + \tan(\theta + i\varphi)\cdot\tan(\theta - i\varphi)}$

(i.e.) $i\tanh(2\varphi) = \dfrac{2i\sin\alpha}{1+1}$

(i.e.) $\tanh(2\varphi) = \dfrac{2\sin\alpha}{1+1}$

(i.e.) $\tanh(2\varphi) = \sin\alpha$

$\Rightarrow\ 2\varphi = \tanh^{-1}(\sin\alpha)$

$$= \log\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)$$

$$= \log\left(\frac{1 + \dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}}{1 - \dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}}\right)$$

$$= \log\left(\frac{1+\tan^2\frac{\alpha}{2} + 2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2} - 2\tan\frac{\alpha}{2}}\right)$$

$$= \log\left(\frac{\left(1+\tan\frac{\alpha}{2}\right)^2}{\left(1-\tan\frac{\alpha}{2}\right)^2}\right)$$

$$= \frac{1}{2}\log\left(\frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}\right)$$

$$= \frac{1}{2}\log\left(\frac{\tan\frac{\pi}{2} + \tan\frac{\alpha}{2}}{1-\tan\frac{\pi}{2}\cdot\tan\frac{\alpha}{2}}\right)$$

$$= \frac{1}{2} \log \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$$

(i.e.) $2\varphi = \frac{1}{2} \log \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$

$\therefore \varphi = \log \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$

This proves (ii)

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Check your progress

Questions :

(1) $2^5 \cosh^6 \theta = \cosh 6\theta + 6\cosh 4\theta + 15\cosh 2\theta + 10$

(2) $32 \sinh^6 \theta = \cosh 6\theta - 6\cosh 4\theta + 15\cosh 2\theta - 10$

10.3 Logarithmic of Complex Numbers

Definition :

If a and b are two complex numbers such that $b = e^a$, then a is called the logarithm of b and we denote $a = \log_e b$.

Example 10.3.1 :

Find logarithm of $x + iy$

Solution : Let $\alpha + i\beta = \log(x + iy)$

Now $\alpha + i\beta = \log(x + iy)$

$\Rightarrow x + iy = e^{\alpha + i\beta}$

$\Rightarrow x + iy = e^{\alpha} e^{i\beta}$

$\Rightarrow x + iy = e^{\alpha}(\cos\beta + i\sin\beta)$

Equating and real part on both sides, we have,

$x = e^{\alpha}\cos\beta$ and $y = e^{\alpha}\sin\beta$

We know that $\sin^2\alpha + \cos^2\alpha = 1$

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(i.e) $\frac{y^2}{e^{2\alpha}}+\frac{x^2}{e^{2\alpha}}=1$

(i.e) $x^2+y^2=e^{2\alpha}$

$\therefore\ 2\alpha=\log(x^2+y^2)$

(i.e) $\alpha=\frac{1}{2}\log(x^2+y^2)$

and $\tan\beta=\frac{y}{x}$

$\Rightarrow\ \beta=\tan^{-1}\left(\frac{y}{x}\right)$

Hence $\log_e(x+iy)=\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\left(\frac{y}{x}\right)$

(i.e) $\log_e(x+iy)=\log r+i\theta$ where $r=\sqrt{x^2+y^2}$ and $\theta=\tan^{-1}\left(\frac{y}{x}\right)$.

Example 10.3.2 :

Find the general value of logarithm of $x+iy$

Solution : Let $\alpha+i\beta=\log(x+iy)$

Now $\alpha+i\beta=\log(x+iy)$

$\Rightarrow\ x+iy=e^{\alpha+i\beta}$

$\Rightarrow\ x+iy=e^{\alpha}e^{i\beta}$

$\Rightarrow\ x+iy=e^{\alpha}(\cos\beta+i\sin\beta)$

$\Rightarrow\ x+iy=e^{\alpha}\{\cos(2n\pi+\beta)+i\sin(2n\pi+\beta)\}$

$\Rightarrow\ x+iy=e^{\alpha}e^{i(2n+\beta)}$

$\Rightarrow\ x+iy=e^{\alpha}e^{i2n+i\beta}$

Thus $Log_e(x+iy)=\alpha+i\beta+2n\pi i$

(i.e) $Log_e(x+iy)=\log_e(x+iy)+2n\pi i$

(i.e) $Log_e(x+iy)=\frac{1}{2}\log(x^2+y^2)+i\left\{\tan^{-1}\left(\frac{y}{x}\right)+2n\pi\right\}$

Hence $Log_e(x+iy)=\frac{1}{2}\log(x^2+y^2)+i\left\{\tan^{-1}\left(\frac{y}{x}\right)+2n\pi\right\}$

(i.e) $Log_e(x+iy)=\log r+i\{\theta+2n\pi\}$ -----------------------------------(10.10)

(0.11)

where $r=\sqrt{x^2+y^2}$ and $\theta=\tan^{-1}\left(\frac{y}{x}\right)$.

Note 1 : If $y=0$, then $\Rightarrow$ $Log_e(x)=\frac{1}{2}\log(x^2)+i(2n\pi)$

(i.e) $Log_e(x)=\log x+i(2n\pi)$

Note 2 : Let $y=0$ and x be negative

Let $x=-x_1$ where $x_1>0$

Now $x=-x_1 = x_1(-1) = x_1(\cos\pi+i\sin\pi)$

Hence $r=x_1$ and $\theta=\pi$

(10.10)$\Rightarrow$ $Log_e(x+iy)=\log r+i\{\theta+2n\pi\}$

(i.e) $Log_e(x)=\log x_1+i(2n\pi+\pi)$

(i.e) $Log_e(x)=\log x_1+i(2n+1)\pi$

Note 3 : Let $x=0$

Now $x+iy = iy$

Hence $r=\sqrt{x^2+y^2}$ and $\theta=\tan^{-1}(\infty)=\frac{\pi}{2}$ -------------------------- (10.12)

(10.12)$\Rightarrow$ $Log_e(iy)=\log r+i\{\theta+2n\pi\}$

(i.e) $Log_e(iy)=\log r+i\left\{\frac{\pi}{2}+2n\pi\right\}$

(i.e) $Log_e(iy)=\log r+i\left\{\frac{1}{2}+2n\right\}\pi$

Space for Hints

Space for Hints

Example 10.3.3 :

Find $Log(1+i)$

Solution :

Let $z = Log(1+i)$

$\therefore\ r = |z| = \sqrt{1+1} = \sqrt{2}$

and $Arg(z) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$$\text{Thus } Log(1+i) = \frac{1}{2}\log r^2 + i(Arg(z) + 2n\pi)$$

$$= \frac{1}{2}\log 2 + i\left(\frac{\pi}{4} + 2n\pi\right)$$

Example 10.3.4 :

Prove that $Log(i) = i(4n+1)\frac{\pi}{2}$

Solution :

Let $z = Log(i)$

$\therefore\ r = |z| = \sqrt{0+1} = 1$

and $Arg(z) = \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$

$$\text{Thus } Log(1+i) = \frac{1}{2}\log r^2 + i(Arg(z) + 2n\pi)$$

$$= \frac{1}{2}\log 2 + i\left(\frac{\pi}{4} + 2n\pi\right)$$

$$\text{Thus } Log(i) = \frac{1}{2}\log r^2 + i(Arg(z) + 2n\pi)$$

$$= \frac{1}{2}\log 1 + i\left(\frac{\pi}{2} + 2n\pi\right)$$

$$= i(4n+1)\frac{\pi}{2}$$

(i.e.) $Log(i) = i(4n+1)\frac{\pi}{2}$.

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Example 10.3.5 :

Find $i^{i} = e^{-(4n+1)\frac{\pi}{2}}$

Solution :

Let $z = i^{i} = e^{i\,Log\,i}$ $(\because\ z^{\omega} = e^{\omega\,Log\,z})$

$$= e^{i\cdot i(4n+1)\frac{\pi}{2}}$$

$$= e^{-(4n+1)\frac{\pi}{2}}$$

Example 10.3.6 :

If $i^{(a+ib)} = a+ib$, prove that $e^{-(4n+1)b\pi}$

Solution :

Given that $a+ib = i^{(a+ib)}$

$$= e^{i\cdot(a+ib)\cdot Log(i)}$$

$$= e^{(a+ib)\cdot i(4n+1)\frac{\pi}{2}}$$

$$= e^{(ai-b)\cdot(4n+1)\frac{\pi}{2}}$$

$$= e^{ia(4n+1)\frac{\pi}{2}}\cdot e^{-b(4n+1)\frac{\pi}{2}}$$

$$= e^{i\theta}\cdot e^{-x} \text{ where } \theta = a(4n+1)\frac{\pi}{2} \text{ and } x = b(4n+1)\frac{\pi}{2}$$

Equating real and imaginary parts on both sides, we get,

$a = e^{-x}\cos\theta$ --(10.13)

and $b = e^{-x}\sin\theta$ --(10.14)

Now $(10.13)^2 + (10.14)^2 \Rightarrow a^2+b^2 = e^{-2x}\cdot(\cos^2\theta+\sin^2\theta)$

(i.e.) $a^2+b^2 = e^{-(4n+1)b\pi}$

This proves the problem.

Example 10.3.7 :

Prove that $Log(1+\cos 2\theta + i\sin 2\theta) = \log(2\cos\theta) + i(2n\pi+\theta)$

Solution :

Let $z = 1+\cos 2\theta + i\sin 2\theta$

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$\therefore \quad |z| = \sqrt{(1+\cos 2\theta)^2 + \sin^2 2\theta}$

$= \sqrt{1+2\cos 2\theta + \cos^2 2\theta + \sin^2 2\theta}$

$= \sqrt{1+2\cos 2\theta + 1}$

$= \sqrt{2(1+\cos 2\theta)}$

$= \sqrt{2 \cdot 2\cos^2\theta)}$

$= 2\cos\theta$

and $Arg(z) = \tan^{-1}\left(\dfrac{\sin 2\theta}{1+\cos 2\theta}\right)$

$= \tan^{-1}\left(\dfrac{2\sin\theta\cos\theta}{2\cos^2\theta}\right)$

$= \tan^{-1}\left(\dfrac{\sin\theta}{\cos\theta}\right)$

$= \tan^{-1}(\tan\theta)$

$= \theta$

$\therefore \; Log(1+\cos 2\theta + i\sin 2\theta) = \dfrac{1}{2}\log r^2 + i(Arg(z) + 2n\pi)$

$= \dfrac{1}{2}\log(2\cos\theta)^2 + i(\theta + 2n\pi)$

(i.e.) $Log(1+\cos 2\theta + i\sin 2\theta) = \log(2\cos\theta) + i(2n\pi + \theta)$

This proves the problem

Example 10.3.8 :

Find $Log_{(-2)}(-3)$

Solution :

Let $x + iy = Log_{(-2)}(-3)$

Now $x + iy = Log_{(-2)}(-3)$

$= \dfrac{Log_e(-3)}{Log_e(-2)}$

(i.e) $(x+iy) \cdot Log_e(-2) = Log_e(-3)$ --------------------------------------(10.15)

Let $z_1 = -3$

$= 3(-1)$

$= 3(\cos\pi + i\sin\pi)$

$\therefore\ |z_1| = 3$ and $Arg(z_1) = \pi$

Again, let $z_2 = -2$

$= 2(-1)$

$= 2(\cos\pi + i\sin\pi)$

$\therefore\ |z_1| = 2$ and $Arg(z_1) = \pi$

Thus $Log(z_1) = Log_e(-3)$

$= \frac{1}{2}\log r^2 + i(2m\pi + Arg(z_1))$

$= \log 3 + i(2m\pi + \pi)$

$= \log 3 + i(2m+1)\pi$

Similarly $Log(z_2) = Log_e(-2)$

$= \frac{1}{2}\log r^2 + i(2n\pi + Arg(z_2))$

$= \log 2 + i(2n\pi + \pi)$

$= \log 2 + i(2n+1)\pi$

Hence (10.15) $\Rightarrow$ $(x+iy)\cdot\{\log 2 + i(2m+1)\pi\} = \log 3 + i(2m+1)\pi$

(i.e) $\{x\log 2 - y(2m+1)\pi\} + i\{x(2m+1)\pi + y\log 2\} = \log 3 + i(2n+1)\pi$

Equating real and imaginary parts on both sides, we get,

$x\log 2 + (2m+1)\pi = \log 3$ ---(10.16)

$y\log 2 + x(2m+1)\pi = (2n+1)\pi$ --- (10.17)

Solving (10.16) and (10.17), we have,

$$x = \frac{\log 2\cdot\log 3 + (2m+1)(2n+1)\pi^2}{((\log 2)^2 + (2n+1)^2\pi^2)}$$

and $y = \dfrac{(2n+1)\pi\log 2 + (2m+1)\pi\log 3}{((\log 2)^2 + (2m+1)^2\pi^2)}$

Check your progress

Questions :

(1) Separate real and imaginary parts of (i) $\tan^{-1}(x+iy)$ and (ii) $\tanh(1+i)$.

Space for Hints

Space for Hints

(2) If $x+iy=\sin(A+iB)$, prove that

$$\text{(i)}\ \frac{x^2}{\cosh^2 B}+\frac{y^2}{\sinh^2 B}=1 \ \text{ and (ii) } \frac{x^2}{\cos^2 A}-\frac{y^2}{\sin^2 A}=1$$

(3) If $\sin(A+iB)=\cos\theta+i\sin\theta$, prove that $\cos^2 A=\sinh^2 B$

10.4 Gregory's Series

Example 10.4.1 :

Prove and prove Gregory's series.

Solution :

Statement :

If $-\frac{\pi}{4}\le\theta\le\frac{\pi}{4}$, then $\theta\ =\ \tan\theta-\frac{\tan^3\theta}{3}+\frac{\tan^5\theta}{5}-\frac{\tan^7\theta}{7}+L$

Proof of the statement :

We know that $e^{i\theta}\ =\ \cos\theta+i\sin\theta$

$$=\ \cos\theta(1+i\tan\theta)$$

Taking log on both sides, we get,

$i\theta\ =\ \log\cos\theta+\log(1+i\tan\theta)$ -- (10.18)

Now $|i\tan\theta|\ =\ |\tan\theta|<1$ provided $-\frac{\pi}{4}\le\theta\le\frac{\pi}{4}$.

Thus (10.18) $\Rightarrow\ i\theta\ =\ \log\cos\theta+\log(1+i\tan\theta)$

$$=\ \log\cos\theta+i\ \log\left(\tan\theta-\frac{\tan^3\theta}{3}+\frac{\tan^5\theta}{5}-\frac{\tan^7\theta}{7}+L\right)$$

Equating imaginary parts on both sides, we get,

$\theta=\tan\theta-\frac{\tan^3\theta}{3}+\frac{\tan^5\theta}{5}-\frac{\tan^7\theta}{7}+L$ --------------------------------- (10.19

which is the required Gregory's series.

Note (1) : Consider the Gregory's series

$\theta=\tan\theta-\frac{\tan^3\theta}{3}+\frac{\tan^5\theta}{5}-\frac{\tan^7\theta}{7}+L$ --------------------------------- (10.2

Space for Hints

Let $x = \tan\theta$

$\therefore\ \theta = \tan^{-1}\theta$

Thus (10.20) $\Rightarrow$ $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + L$

Note (2) :

Put $\theta = \dfrac{\pi}{4}$ in the Gregory's series (10.19), we have,

$$\frac{\pi}{4} = \tan\left(\frac{\pi}{4}\right) - \frac{1}{3}\tan^3\left(\frac{\pi}{4}\right) + \frac{1}{5}\tan^5\left(\frac{\pi}{4}\right) - \frac{1}{7}\tan^7\left(\frac{\pi}{4}\right) + L$$

(i.e) $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + L$ -- (10.21)

Further (10.21) can be modified as $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + L$

$$\frac{\pi}{4} = 1 - \left(\frac{1}{3} - \frac{1}{5}\right) - \left(\frac{1}{7} - \frac{1}{9}\right) - L$$

(i.e) $\dfrac{\pi}{4} = 1 - 2\left(\dfrac{1}{3\cdot 5} + \dfrac{1}{7\cdot 9} + \dfrac{1}{11\cdot 13} L\ \right).$

Example 10.4.2 :

Using Gregory's series, find $1 - \dfrac{1}{3\cdot 4^2} + \dfrac{1}{5\cdot 4^4} + \dfrac{1}{7\cdot 4^6} L$

Solution :

Now $1 - \dfrac{1}{3\cdot 4^2} + \dfrac{1}{5\cdot 4^4} + \dfrac{1}{7\cdot 4^6} L$

$$= 4\left[\frac{1}{4} - \frac{1}{3\cdot 4^3} + \frac{1}{5\cdot 4^5} + \frac{1}{7\cdot 4^7} L\ \right]$$

$$= 4\left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + L\ \right] \text{ where } x = \frac{1}{4}$$

$$= 4\cdot\tan^{-1} x$$

$$= 4\cdot\tan^{-1}\left(\frac{1}{4}\right)$$

Thus $1 - \dfrac{1}{3\cdot 4^2} + \dfrac{1}{5\cdot 4^4} + \dfrac{1}{7\cdot 4^6} L\ = 4\cdot\tan^{-1}\left(\dfrac{1}{4}\right).$

Space for Hints

Example 10.4.2 :

Find $\left(\frac{2}{3}+\frac{1}{7}\right)-\frac{1}{3}\left(\frac{2}{3^3}+\frac{1}{7^3}\right)+\frac{1}{5}\left(\frac{2}{3^5}+\frac{1}{7^5}\right)-\frac{1}{7}\left(\frac{2}{3^7}+\frac{1}{7^7}\right)+L$

Solution :

Let $S = \left(\frac{2}{3}+\frac{1}{7}\right)-\frac{1}{3}\left(\frac{2}{3^3}+\frac{1}{7^3}\right)+\frac{1}{5}\left(\frac{2}{3^5}+\frac{1}{7^5}\right)-\frac{1}{7}\left(\frac{2}{3^7}+\frac{1}{7^7}\right)+L$

$$\therefore S = 2\left[\frac{1}{3}-\frac{1}{3}\left(\frac{1}{3^3}\right)-\frac{1}{5}\left(\frac{1}{3^5}\right)-\frac{1}{7}\left(\frac{1}{3^7}\right)L\right]+\left[\frac{1}{7}-\frac{1}{3}\left(\frac{1}{7^3}\right)-\frac{1}{5}\left(\frac{1}{7^5}\right)-\frac{1}{7}\left(\frac{1}{7^7}\right)L\right]$$

$$\text{(i.e) } S = 2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right) \quad \text{--} \quad (10.22)$$

$$\text{Now } 2\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{3}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{3}}{1-\frac{1}{3}\cdot\frac{1}{3}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{2}{3}}{\frac{8}{9}}\right)$$

$$= \tan^{-1}\left(\frac{3}{4}\right)$$

$$\therefore (10.22) \Rightarrow S = \tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{25}{28}}{\frac{25}{28}}\right)$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

Space for Hints

Thus $\left(\frac{2}{3}+\frac{1}{7}\right)-\frac{1}{3}\left(\frac{2}{3^3}+\frac{1}{7^3}\right)+\frac{1}{5}\left(\frac{2}{3^5}+\frac{1}{7^5}\right)-\frac{1}{7}\left(\frac{2}{3^7}+\frac{1}{7^7}\right)+\text{L} \;=\; \frac{\pi}{4}$

Example 10.4.3 :

Prove that $\tan^{-1}x=\frac{\pi}{4}+\left(\frac{x-1}{x+1}\right)-\frac{1}{3}\left(\frac{x-1}{x+1}\right)^3+\frac{1}{5}\left(\frac{x-1}{x+1}\right)^5-\frac{1}{7}\left(\frac{x-1}{x+1}\right)^7+\text{L}$

Proof :

$$\text{RHS}=\frac{\pi}{4}+\left(\frac{x-1}{x+1}\right)-\frac{1}{3}\left(\frac{x-1}{x+1}\right)^3+\frac{1}{5}\left(\frac{x-1}{x+1}\right)^5-\frac{1}{7}\left(\frac{x-1}{x+1}\right)^7+\text{L}$$

$$=\frac{\pi}{4}+y-\frac{1}{3}y^3+\frac{1}{5}y^5-\frac{1}{7}y^7+\text{L} \quad \text{where } y=\frac{x-1}{x+1}$$

$$=\tan^{-1}(1)+\tan^{-1}y$$

$$=\tan^{-1}\left(\frac{1+y}{1-y}\right)$$

$$=\tan^{-1}x$$

$$=\text{LHS}$$

Thus $\tan^{-1}x=\frac{\pi}{4}+\left(\frac{x-1}{x+1}\right)-\frac{1}{3}\left(\frac{x-1}{x+1}\right)^3+\frac{1}{5}\left(\frac{x-1}{x+1}\right)^5-\frac{1}{7}\left(\frac{x-1}{x+1}\right)^7+\text{L}$

Example 10.4.4 :

If $|x|<\sqrt{2}-1$, prove that

$$2\left[x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\text{L}\right]=\frac{2x}{1-x^2}-\frac{1}{3}\left(\frac{2x}{1-x^2}\right)^3+\frac{1}{5}\left(\frac{2x}{1-x^2}\right)^5-\text{L}$$

Proof :

Let $y=\frac{2x}{1-x^2}$

Now $|y|<1$

$\Rightarrow \left|\frac{2x}{1-x^2}\right|<1$

$\Rightarrow -1<\frac{2x}{1-x^2}<1$

$\Rightarrow |x|<\sqrt{2}-1$

Space for Hints

Now RHS $= \dfrac{2x}{1-x^2} - \dfrac{1}{3}\left(\dfrac{2x}{1-x^2}\right)^3 + \dfrac{1}{5}\left(\dfrac{2x}{1-x^2}\right)^5 - L$

$$= y - \frac{1}{3}y^3 + \frac{1}{5}y^5 - \frac{1}{7}y^7 + L$$

$$= \tan^{-1} y$$

$$= \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$= \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) \text{ where } x = \tan\theta$$

$$= \tan^{-1}(\tan 2\theta)$$

$$= 2\theta$$

$$= 2\tan^{-1} x$$

$$= 2\left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + L\right]$$

Thus $2\left[x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + L\right] = \dfrac{2x}{1-x^2} - \dfrac{1}{3}\left(\dfrac{2x}{1-x^2}\right)^3 + \dfrac{1}{5}\left(\dfrac{2x}{1-x^2}\right)^5 - L$

This proves the problem

Example 10.4.5 :

If $-\dfrac{\pi}{4} < x < \dfrac{\pi}{4}$ then prove that

$$\tan x - \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x - L = \tanh x + \frac{1}{3}\tanh^3 x + \frac{1}{5}\tanh^5 x + L$$

Proof :

LHS $\tan x - \dfrac{1}{3}\tan^3 x + \dfrac{1}{5}\tan^5 x - L$

$$= \tan^{-1}(\tan x) = x \text{ -- } (10.23)$$

and RHS $= \tanh x + \dfrac{1}{3}\tanh^3 x + \dfrac{1}{5}\tanh^5 x + L$

$$= y + \frac{1}{3}y^3 + \frac{1}{5}y^5 + L \quad \text{where } y = \tanh x$$

$$= \tanh^{-1} y$$

Space for Hints

$$= \tanh^{-1}(\tanh x)$$

$$= x \text{ -- } (10.24)$$

From (10.23) and (10.24), we have,

$$\tan x - \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x - \text{L} = \tanh x + \frac{1}{3}\tanh^3 x + \frac{1}{5}\tanh^5 x + \text{L}$$

This proves the problem.

Check your progress

Questions :

(1) Prove that $\pi = 2\sqrt{3}\left[1 - \frac{1}{3^2} + \frac{1}{5}\cdot\frac{1}{3^2} - \frac{1}{7}\cdot\frac{1}{3^3} + \text{L}\right]$

(2) Prove that $\frac{1}{2^3} - \frac{1}{3\cdot 2^7} + \frac{1}{5\cdot 2^{11}} - + \text{L} = \frac{1}{2}\tan^{-1}\left(\frac{1}{4}\right)$

(3) Prove that $\left(1 - 3^{-1/2}\right) - \frac{1}{3}\left(1 - 3^{-3/2}\right) + \frac{1}{5}\left(1 - 3^{-5/2}\right) - \text{L} = \frac{\pi}{12}$.

Summary

In this unit we have learned that the nature hyperbolic functions, inverse hyperbolic functions and how to find the logarithmic of a complex number and Gregory's series

Further Reading

You can also refer the following books for further reading.

(1) Trigonometry by T.K.Manicavachagom Pillai and others

(2) Trigonometry by Arumugam and others.